Towards the Construction of Local Logarithmic Conformal Field Theories

Diplomarbeit

angefertigt am

Institut für Theoretische Physik
der Leibniz Universität Hannover

von

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Hannover, im Oktober 2007

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Abstract

In this thesis first steps are taken towards the construction of local logarithmic conformal field theories (LCFTs). After resuming some of the central concepts and definitions of conformal field theory we explore local non-chiral LCFTs whose chiral halves possess arbitrary Jordan rank. Our approach rests upon the analysis of the conformal symmetry and aims for comprehensive generality. Beside some rather general assumptions on the structure of a chiral LCFT the deduced statements about its local non-chiral equivalent can be called generic.

In the first part of this thesis we investigate the field content of local LCFTs and present two methods how to construct local representations of the symmetry algebra as subrepresentations of the tensor product of chiral and anti-chiral Jordan cells. The gained Hilbert spaces are not identical but isomorphic. The symplectic fermion modell suggests one of the solutions to be more intuitive.

In the second part we study the possibility to assemble generic chiral correlation functions to local correlation functions of the corresponding non-chiral theory. Beside the constraint of locality, invariance under the global conformal group, duality and monodromy invariance have to be implemented. We propose a constructive method, discuss the generality of our solution and check its consistency with previous findings.

Zusammenfassung

Diese Arbeit befasst sich mit grundlegenden Schritten in Richtung einer Konstruktion nicht chiraler, lokaler logarithmisch konformer Feldtheorien (LCFTs). Nach kurzer Darstellung einiger der wichtigsten Definitionen und Relationen konformer Feldtheorie werden Aussagen über lokale LCFTs entwickelt, deren chirale Hälften beliebigen Jordanrang aufweisen. Die gewählte Herangehensweise basiert auf der Ausschöpfung der konformen Symmetriealgebra und zielt darauf ab, Aussagen von weitgehender Allgemeingültigkeit zu treffen. Abgesehen von wenigen schwachen Annahmen in Bezug auf die Struktur chiraler Theorien bleibt die Untersuchung der entsprechenden nicht chiralen Theorien generisch.

Im ersten Teil der Arbeit analysieren wir den Feldinhalt lokaler LCFTs und geben zwei Methoden an, lokale Darstellungen der Symmetriealgebra als Unterraum des Tensorprodukts von chiralen und antichiralen Jordanzellen zu konstruieren. Die nach beiden Methoden gewonnenen Hilberträume sind nicht identisch, jedoch isomorph. Das symplektische Fermionen–Modell zeichnet schließlich eine der Lösungen als die intuitivere aus.

Im zweiten Teil der Arbeit wird die Möglichkeit untersucht, generische chirale Korrelationsfunktionen zu lokalen Korrelatoren der korrespondierenden nicht chiralen Theorie zu kombinieren. Neben der Lokalitätsbedingung ist die Invarianz unter der global konformen Gruppe sowie Dualität und Monodromieinvarianz der Korrelatoren zu gewährleisten. Wir stellen eine Methode vor, die die Konstruktion der lokalen Korrelatoren erlaubt, diskutieren die Allgemeinheit der Lösung und prüfen die Konsistenz der Voraussagen der vorgestellten Methode mit bisherigen Erkenntnissen.

Contents

1 Introduction

Conformal quantum field theories (CFTs) in two dimensions exhibit an exceedingly rich structure. The symmetry algebra of these theories is infinite-dimensional and decomposes in two independent sectors of opposite chirality. The decoupling of the chiral and anti-chiral "half" gives reason to a very elegant and beneficial technique: Many statements about CFTs can be derived considering only one sector, either the chiral or the anti-chiral. Due to the infinite number of local symmetries the adherent calculations are under certain circumstances exactly solvable. Results of one sector can be transferred to the other one by mirroring the chirality. Subsequently both halves have to be glued together to form the full, non-chiral theory.

The first to realize the abovementioned chances were Belavin, Polyakov and Zamolodchikov [1]. Their work opened up a revolutionary since non-pertubative approach to a subclass of quantum field theories. In the course of the following investigation of these theories it was discovered that CFT correlation functions may exhibit logarithmic divergencies. The circumstances of their occurrence were explored [2] and it turned out that non-logarithmic CFT is actually a special case of the more general concept of logarithmic CFT (LCFT). Since this key finding, an enormous amount of work was done to evolve LCFTs. Their applications to other fields in physics, for example to the theory of percolation and critical disordered systems have been explored. Many structural aspects have been studied in detail. In particular the powerfull techniques from non-logarithmic CFT have been ported to LCFT, especially those that do exclusively use chiral informations: Correlation functions are calculated in [3–12] , fusion rules are investigated in [13–17] and null vectors in [18, 19].

All cited findings were stated for chiral theories only. Enlarging their scope to non-chiral theories and implementing the constraints of locality is a task of elementary importance, as only local theories have physical interpretation. Unfortunately, the construction of local theories turned out to be non-trivial: Unlike conventional, non-logarithmic CFTs, LCFTs were found to be non-factorisable. Insofar speaking about chiral halves in the context of LCFTs might be delusive. As we will see, assembling a local theory out of a chiral LCFT and its anti-chiral counterpart is evocative of screwing them into each other rather than of combining two halves. Anyhow, we will stick with the familiar naming convention and refer in abuse of language to the chiral theories as chiral halves.

So far, only few attempts have been made to close the gap between the well-known chiral and the almost unknown non-chiral LCFTs. Gaberdiel and Kausch suc-

ceeded in constructing a non-chiral local theory at $c = -2$ by solving the conformal bootstrap [20]. This construction was interpreted in terms of symplectic fermions [21] and enabled a detailed comparison of the two-dimensional Abelian sandpile model with the local triplet theory at $c = -2$ [22]. The logarithmic triplet theory with boundary was studied in [23] for $c_{1,2} = -2$ and the generalisation to rational $c_{q,p}$ models was attempted in [24]. Stating remarkable structural similarities between the local triplet theory and supergroup WZNW models, SCHOMERUS et al. suggested that consulting the better understood local WZNW models might promote the construction of generic local LCFTs. Local logarithmic bulk correlation functions for the $GL(1|1)$ WZNW model have been computed in [25], non-chiral indecomposable representations on which the zero-mode of the energy-momentum tensor is not digonalizable were investigated by means of the WZNW model on the supergroup $PSU(1,1|2)$ [26]. A rather general discussion on conclusions for local LCFTs from the supergroup WZNW point of view can be found in [27].

In this thesis, we will choose an approach which rests solely upon the analysis of the conformal symmetry and aims for comprehensive generality. Beside some rather general assumptions on the structure of an chiral LCFT the deduced predictions about its local non-chiral equivalent can be called generic.

This thesis is organized as follows: In **Section 2** we present general foundations of conformal field theories and logarithmic conformal field theories. This introduction should enable the reader to understand the suceeding chapters. Furthermore we fix the naming convention and introduce the notation used.

In **Section 3** the aspect of locality is explored and an algebraic formulation of the locality constraint is derived.

Section 4 shows in detail why in case of indecomposable representations the non-chiral representation is not the tensor produt of its left- and right chiral equivalent, but only a quotientspace thereof. The chiral concept of Jordan levels is enlarged to the non-chiral case. Two methods are presented how to construct the space of states of a local LCFT. Both methods predict a non-chiral local theory to possess the same rank as the halves it is composed of. In other respects the results of both methods turn out to be mutually excluding. The consideration of a specific model suggests one of the proposed methods to be more intuitive.

From **Section 5** we turn our attention to LCFT correlation functions. After a brief recapitulation about generic chiral correlation functions, we explore the possibility to compose generic non-chiral correlators out of generic chiral ones. Beside the constraint of locality, invariance under the global conformal group, duality and monodromy invariance have to be implemented. We propose a construction method,

discuss the generality of our solution and check its consistency with previous findings.

Technical details of our calculations can be found in the included appendices.

2 Preliminaries

Locally scale invariant quantum field theories are called conformal field theories. In this section a basic familiarity with those concepts this thesis rests upon shall be achieved and conventions as well as matters of notation shall be introduced. In the majority of cases we will skip extensive calculations and proofs. The reader unfamiliar with CFT may be refered to [28, 29, 31] for an elaborated introduction. A more mathematical approach is chosen in [32]. For a detailed description of Virasoro algebras see [33, 34]. Foundations of LCFTs can be found in [14] and [35].

We will start with an investigation of the conformal symmetry in a setting of flat d-dimensional spacetime $\mathbb{R}^{p,q}$, with $d = p + q$ and signature $((-1)^p, (+1)^q)$. From subsection (2.1.3) we restrict ourselves on two-dimensional systems. As we will see $d = 2$ conformal field theories provide a specific attraction: in two spacetime dimensions the conformal symmetry algebra is infinite-dimensional and manifests a rich structure, the Virasoro algebra.

Logarithmic conformal field theory is a generalisation of conformal field theory. From section (2.2) we will focus on the question, how the concepts of CFT have to be modified in the framework of LCFT.

2.1 Conformal Field Theory

2.1.1 Conformal transformations

Conformal transformations possess the characteristic trait to preserve angles between two arbitrary curves on a manifold. This can be expressed in terms of the effect of coordinate transformations on the metric tensor: Arbitrary coordinate transformations $f : x \to x'$ act on the metric as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x). \tag{1}$$

A transformation f is called conformal if it leaves the metric tensor invariant up to a position-depending scale-factor, i. e. if the left hand side of (1) is proprotional to $g_{\mu\nu}(x)$:

$$\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = \Omega(x) g_{\mu\nu}(x). \tag{2}$$

2.1.2 Conformal group and algebra

Infinitesimal transformations $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ altering the metric tensor as demanded above satisfy a constraint which is known as the conformal Killing equation:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \tfrac{2}{d}\, \partial \cdot \epsilon \; g_{\mu\nu}, \qquad \partial \cdot \epsilon \equiv \partial^\lambda \epsilon_\lambda. \tag{3}$$

Contracting equation (3) first with $\partial_\rho \partial^\nu$ then with $\partial^\rho \partial_\nu$ and adding both equations one finds:

$$\left(1 - \tfrac{2}{d}\right) \partial_\rho \partial_\mu \partial \cdot \epsilon = 0. \tag{4}$$

For $d > 2$ it follows immediately that ϵ^μ can be of at most second order in the coordinates x^μ:

$$\epsilon^\mu(x) = \alpha^\mu + \beta^\mu_\nu x^\nu + \gamma^\mu_{\nu\rho} x^\nu x^\rho. \tag{5}$$

The conformal transformations thus determined form a group. Its $\frac{1}{2}(d+1)(d+2)$ generators

$$D = x^\mu \partial_\mu \tag{6a}$$
$$P_\mu = \partial_\mu \tag{6b}$$
$$M_{\mu\nu} = \tfrac{1}{2}\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \tag{6c}$$
$$K_\mu = x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu \tag{6d}$$

can be identified as the generators of rotations, translations, dilations and so called special conformal transformations. One can write out the commutation rules of the generators to see that they form a closed algebra. The component containing the identity is isomorphic to $SO(p+1, q+1)$.

2.1.3 Conformal symmetry in two dimensions

In two dimensions the restriction that $\epsilon(x)$ is of at most second order in x drops. Subsequently we will address ourselves to the two-dimensional manifold $\mathbb{R}^{1,1}$ with the flat metric tensor $\eta_{\mu\nu}$. In the course of Wick rotation the Minkowski metric can be replaced by $\delta_{\mu\nu}$. Moreover the time coordinate becomes imaginary: $x^0 \to -\mathrm{i}x^2$. Therewith (3) turns out to be identical to the Cauchy-Riemann differential equations:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \,, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{7}$$

With complex variables,

$$z = x_1 + \mathrm{i}x_2 \,, \qquad \bar{z} = x_1 - \mathrm{i}x_2 \tag{8}$$

the constraints (7) determine an infinititesimal coordinate transformation

$$z \mapsto z' = z + \epsilon(z,\bar{z}) \ , \qquad \bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\epsilon}(z,\bar{z}) \tag{9}$$

to be conformal if

$$\partial_{\bar{z}}\epsilon(z,\bar{z}) = 0 \ , \qquad \partial_z\bar{\epsilon}(z,\bar{z}) = 0. \tag{10}$$

That is: any holomorphic function $\epsilon(z)$ defines an infinitesimal conformal transformation (9), any anti-holomorphic function $\bar{\epsilon}(\bar{z})$, too. Expanding the holomorphic (and anti-holomorphic) transformations in Laurent series $\epsilon(z) = \sum_n \epsilon_{-n} z^n$ we are led to introduce the generators of the local conformal transformations:

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \qquad n \in \mathbb{Z} \tag{11}$$

Calling up the association of helicity the sector that depends on the coordinate z only is often referred to as the chiral sector. The scope of $\bar{z}$ is named anti-chiral. The chiral and anti-chiral conformal generators decouple and satisfy mutually independently a Lie algebra

$$\left[l_n, l_m\right] = (m-n)\, l_{m+n} \ , \qquad \left[\bar{l}_n, \bar{l}_m\right] = (m-n)\, \bar{l}_{m+n} \tag{12}$$

the so-called Witt algebra.

As mentioned before it is in many respects convenient to treat z and $\bar{z}$ as linearly independent and consider the chiral and anti-chiral sector separately. This proceeding entails an artificial doubling of variables which finally has to be removed by reimplementing the reality condition $z^* = \bar{z}$.

Though the conformal algebra is infinite-dimensional, not every generator is well-defined globally. The (global) conformal group is generated by the sub-algebra $\{l_n, \bar{l}_n\}$ for $n = -1, 0, 1$ and is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2 \times SL(2,\mathbb{C})/\mathbb{Z}_2$. The corresponding global conformal transformations are known as Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \ , \quad \bar{z} \mapsto \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}. \tag{13}$$

2.1.4 Virasoro algebra

In a classical theory, the generators of a Lie algebra may be defined such that they satisfy the Poisson bracket relations

$$\left[t^a, t^b\right]_{P.B.} = f_c^{ab} t^c. \tag{14}$$

Following Dirac's quantization procedure the correspondig quantum commutation is

$$\left[t^a, t^b\right] = \mathrm{i}\hbar f_c^{ab} t^c + \mathcal{O}(\hbar^2) \tag{15}$$

The terms of order $\hbar^2$ are called central extension of the algebra (15) and are constrained by Jacobi identities. The simplest realisation to consider is that the extensions $\mathcal{O}\left(\hbar^2\right)$ are complex numbers which satisfy

$$c_{a,b} = \frac{c}{12}\, a\left(a^2 - 1\right) \delta_{a,-b}\ . \tag{16}$$

Therewith the chiral and anti-chiral symmetry algebra of a conformal quantum field theory in two dimensions reads

$$\begin{aligned} \left[L_n, L_m\right] &= (n-m)\, L_{n+m} + \tfrac{c}{12}\left(n^3 - n\right) \delta_{n,-m} \\ \left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\, \bar{L}_{n+m} + \tfrac{\bar{c}}{12}\left(n^3 - n\right) \delta_{n,-m} \\ \left[L_n, \bar{L}_m\right] &= 0. \end{aligned} \tag{17}$$

This is the famous Virasoro algebra, a class of algebras parametrized by the central charge c. Note that in perfect analogy to the Witt algebra the chiral and the anti-chiral part decouple. Moreover, the subalgebra generated by $\left\{L_n, \bar{L}_n\right\}$, $n = -1, 0, 1$ is not affected by the central charge, i. e. the global conformal group continues to be isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2 \times SL(2,\mathbb{C})/\mathbb{Z}_2$.

The central extension (16) of the symmetry algebra eliminates the possibility of the vacuums having full symmetry. The maximal symmetry we can claim without violating the algebra is

$$L_n \left|0\right\rangle = \bar{L}_n \left|0\right\rangle = 0, \quad \text{for } n \geq -1. \tag{18}$$

As equation (18) in particular implies invariance under the global conformal group $\left|0\right\rangle$ is often referred to as $SL(2,\mathbb{C})$ invariant vacuum. The state $\left|0\right\rangle$ is unique. Its hermitean conjugate $\left\langle 0\right|$ satisfies the relation

$$\left\langle 0\right| L_n = \left\langle 0\right| \bar{L}_n = 0, \quad \text{for } n \leq 1. \tag{19}$$

2.1.5 Virasoro representation theory

A heighest weight representation module of the Virasoro algebra is built up on a primary field. The primary and with it the representation are characterized by two real parameters h and $\bar{h}$, the chiral and anti-chiral conformal weights. A field

$\phi_{h,\bar{h}}$ is named a primary field if it transforms under any conformal transformation $z \mapsto w(z)$ ($\bar{z} \mapsto \bar{w}(\bar{z})$ respectively) as

$$\phi_{h,\bar{h}}(z,\bar{z}) \mapsto \phi'_{h,\bar{h}}(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi_{h,\bar{h}}(z,\bar{z}). \tag{20}$$

From equation (20) follows that an infinitesimal transformation $z \mapsto w = z + \epsilon_n z^{n+1}$ induced through a conformal generator L_n causes the following variation of a primary field:

$$\delta_n \phi_h(z) = \left[L_n, \phi_h(z)\right] = \left(z^{n+1}\partial + h\,(n+1)\,z^n\right) \phi_h(z). \tag{21}$$

Here, for the first time we used the chance to omit one half of the theory. The definitive transformation property (20) of a primary field is irrespective of one coordinate to be suppressed. It thus remains meaningful to speak of a chiral primary field $\phi_h(z)$.

Fields that do not transform according to equation (20) are referred to as secondary fields. Typical examples for secondary fields are derivatives of primary fields. If a field satisfies the property (20) only with respect to the global conformal group, it is called quasi-primary.

A field–state isomorphism

$$|h\rangle := \lim_{z\to 0} \phi_h(z)\,|0\rangle \tag{22}$$

allows us to study Virasoro representation theory in terms of states. Highest weight states are generated from the vacuum by primary fields. They are eigenstates of the Virasoro zero modes with eigenvalue h:

$$L_0\,|h\rangle = h\,|h\rangle. \tag{23}$$

Acting with L_n on the state $|h\rangle$ decreases its L_0 eigenvalue by n:

$$L_0\,L_n\,|h\rangle = \big(L_n\,L_0 - [L_n, L_0]\big)|h\rangle = (h-n)\,L_n\,|h\rangle. \tag{24}$$

A highest weight state defines the state within a representation that exhibits the lowest L_0 eigenvalue. From equation (24) one can conclude that it is annihilated by all positive generators:

$$L_n\,|h\rangle = 0, \quad \text{for } n > 0. \tag{25}$$

Negative Virasoro modes L_n, $n \leq -1$ can be used to generate other states of the representation, the so-called descendant states.

2.1.6 Ward Identities and Correlation functions

Let $\langle \phi_1 \dots \phi_n \rangle$ be an n-point correlation function

$$\langle \phi_1 \dots \phi_n \rangle := \langle 0 \,|\, \phi_1(z_1) \dots \phi_n(z_n) \,|\, 0 \rangle \,, \tag{26}$$

where any ϕ_i denotes a primary field of conformal weight h_i. According to the relations (18) and (19) $\langle 0 \,|\, L_q = L_q | 0 \rangle = 0$ for $q = -1, 0, 1$. Hence, for these values of q it follows that

$$\begin{aligned} 0 &= \langle 0 \,|\, L_q \, \phi_1 \dots \phi_n \,|\, 0 \rangle \\ &= \sum_i \langle 0 \,|\, \phi_1 \dots \phi_{i-1} \left[L_q, \phi_i \right] \phi_{i+1} \dots \phi_n \,|\, 0 \rangle + \langle 0 \,|\, \phi_1 \dots \phi_n \, L_q \,|\, 0 \rangle \,. \end{aligned} \tag{27}$$

The last terms vanishes. Therewith we derived the so called global conformal Ward identities (GCWIs)

$$\begin{aligned} \mathcal{L}_q \langle \phi_1 \dots \phi_n \rangle &:= \sum_i \langle 0 \,|\, \phi_1 \dots \phi_{i-1} \left[L_q, \phi_i \right] \phi_{i+1} \dots \phi_n \,|\, 0 \rangle \\ &= \sum_{i=1}^{n} z_i^q \left[z_i \partial_i + (q+1) h_i \right] \langle \phi_1 \dots \phi_n \rangle = 0 \qquad \text{for } q = -1, 0, 1. \end{aligned} \tag{28}$$

The second identity results from equation (21).

The GCWIs enable us to fix the generic structure of an n-point function up to an a priori undetermined structure function of $n-3$ $SL(2,\mathbb{C})$ invariant crossratios. Thus for example the two-point function reads

$$\langle \phi_1 \, \phi_2 \rangle = \delta_{h_1, h_2} \frac{C\,(h_1, h_2)}{z_{12}^{h_1+h_2}} \,, \tag{29}$$

where the usual abbreviation $z_{ij} := z_i - z_j$ is used. The three-point functions is given by

$$\langle \phi_1 \, \phi_2 \, \phi_3 \rangle = \frac{C\,(h_1, h_2, h_3)}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}}. \tag{30}$$

The four-point function is known to be of the form

$$\langle \phi_1 \, \phi_2 \, \phi_3 \, \phi_4 \rangle = \prod_{i<j} z_{ij}^{\mu_{ij}} F(x), \quad \sum_{i \neq j} \mu_{ij} = -2h_j. \tag{31}$$

where $F(x)$ is the aforementioned undetermined structure function of the crossratio $x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$. Beyond the sketched features, CFT correlators are very well-elaborated. Structure functions may be subject to further determination due to additional local symmetries. Correlation functions of descendant fields can be traced back to those of primary fields. In the scope of this thesis we will be concerned with neither of these cases.

2.2 Logarithmic Conformal Field Theory

2.2.1 Indecomposable Virasoro representations

Logarithmic conformal field theories feature indecomposable but reducible representations of the chiral symmetry algebra. In this thesis we will consider only the case where such indecomposable representations occur with respect to the Virasoro zero modes. Through appropriate choice of the basis, the generators L_0 and $\overline{L_0}$ can be transformed into Jordan normal form. A rank r Jordan cell is spanned by r fields $\{\Psi_{(h,r-1)}, \ldots, \Psi_{(h,1)}, \Psi_{(h,0)}\}$. The action of the zero mode L_0 of the Virasoro algebra is then given by

$$L_0 \Psi_{(h,k)}(0)\,|\,0\rangle = h\,\Psi_{(h,k)}(0)\,|\,0\rangle + (1-\delta_{k,0})\Psi_{(h,k-1)}(0)\,|\,0\rangle, \tag{32}$$

where h as usual denotes the conformal weight. The parameter k grades the fields within the Jordan cell and will be termed Jordan level. The field with level $k = 0$ is an ordinary primary field. All other fields $\Psi_{(h,k)}$ with $k \geq 1$ are called logarithmic (partner) fields. The adjunct "logarithmic" refers to the fact that inserting these fields in correlators gives rise to logarithmic terms.

2.2.2 Naming convention

Subsequently we are forced to sharpen the concept of primary fields: In an LCFT the operator product expansion of two primary fields with non-integer conformal weight might produce logarithmic fields on the right hand side. These primaries are called pre-logarithmic or twist fields. Primary fields whose operator product expansion among each other does merely contain primary fields and its descendants but no logarithmic fields are called proper-primaries. In the scope of this work, only reducible representations are regarded whose irreducible subrepresentations $\Psi_{(h_i,0)}(z_i)$ accord to proper primaries. Furthermore, we assume the logarithmic partner fields of the proper primary to be quasi-primary, i. e.

$$L_n \Psi_{(h_i,k_i)}(0)\;|\,0\rangle = 0 \quad \forall n \geq 1. \tag{33}$$

If an LCFT accomodates more than one indecomposable representation one can consider every Jordan cell to be of rank $r(h) = r$, with r being the rank of the largest Jordan cell. Hypothetically emerging smaller Jordan cells can be padded with fields that are to be set zero afterwards.

The following decomposition of the non-diagonal action of the Virasoro modes on LCFT n-point correlators will turn out to be advantageous:

$$\mathcal{L}_q \langle \dots \rangle = \left[O_q + \sum_{i=1}^{n} z_i^q (q+1) \delta_{h_i} \right] \langle \dots \rangle , \tag{34}$$

where O_q abbreviates the diagonal part of the action as known from ordinary non-logarithmic CFT

$$O_q \langle \dots \rangle = \sum_{i=1}^{n} z_i^q \left[z_i \partial_i + (q+1) h_i \right] \langle \dots \rangle .$$

The off-diagonal, nilpotent part is generated by operators δ_{h_i}. They act on a logarithmic field by reducing its Jordan level by one, on a primary by annihilating the field:

$$\delta_{h_j} \Psi_{(h_i,k_i)}(z_i) = \delta_{ij} \left(1 - \delta_{0,k_i}\right) \Psi_{(h_i,k_{i-1})}(z_i). \tag{35}$$

Below we will augment the introduced glossary with entities marked with a bar. These can be obtained from those without a bar by complex conjugation of all variables z and providing all parameters h, k and g with a superscript line. For h, k and g this line is not related to complex conjugation but only indicates that parameters of the anti-chiral theory are denoted.

To distiguish quantities of a full local theory from those living on its right- or left-handed half we will apply the pair of concepts “non-chiral” - “chiral”. If used in this sense the latter shall cover anti-chiral quantities, too.

2.2.3 Vanishing correlators

A first look at LCFT n-point functions exposes a curiosity which will gain some relevancy in the following. Let $\Psi_{(h,k)}(z)$ be a field from a rank r Jordan cell. Translation invariance of the one-point function enforces $\langle \Psi_{(h,k)} \rangle$ to be constant:

$$\mathcal{L}_{-1} \langle \Psi_{(h,k)} \rangle = 0 \;\Rightarrow\; \langle \Psi_{(h,k)} \rangle = const. \tag{36}$$

Moreover, the GCWI for $q = 0$ imposes the following restriction (cf. (34)):

$$\mathcal{L}_0 \langle \Psi_{(h,k)} \rangle = z \partial_z \langle \Psi_{(h,k)} \rangle + h \langle \Psi_{(h,k)} \rangle + (1 - \delta_{0,k}) \langle \Psi_{(h,k-1)} \rangle = 0. \tag{37}$$

The first term vanishes due to (36). As a consequence the only field with non-vanishing vacuum expectation value (vev) is the field with $h = 0$ and highest Jordan

level $r-1$. This means in particular that the one-point function of proper primaries residing in a non-trivial Jordan cell is zero.

An analogous argumentation for higher n-point functions yields the following result [4]: Correlators which contain fields from rank r Jordan cells are non-zero only if the sum over the Jordan levels is greater or equal $r-1$:

$$\left\langle \Psi_{(h_1,k_1)} \dots \Psi_{(h_n,k_n)} \right\rangle \neq 0 \;\Leftrightarrow\; K := \sum_{i=1}^{n} k_i \geq r-1. \tag{38}$$

This is especially true for the Shapovalov form $\langle h, k \,|\, h', k' \rangle$ [35] as it corresponds to the two-point function $\left\langle \Psi_{(h,k)}(z_1) \Psi_{(h',k')}(z_2) \right\rangle$ evaluated at $z_1 = \infty$, $z_2 = 0$.

3 Locality constraints

The fundamental postulation on a non-chiral theory is locality of the fields or, in terms of correlation functions, singlevaluedness of the amplitudes. We will follow the argumentation of [20] deriving the resulting equations of constraints. Evaluating equation (34) for $q = 1$ and $n = 2$ yields:

$$O = [z\partial_z + h_1 + h_2 + \delta_{h_1} + \delta_{h_2}] \left\langle \Psi_{(h_1,\bar{h}_1,\mathbf{k_1})}(z,\overline{z})\, \Psi_{(h_2,\bar{h}_2,\mathbf{k_2})}(0,\overline{0}) \right\rangle . \tag{39}$$

An analogous identity is satisfied for the anti-chiral Virasoro zero mode. By integrating the difference of the two differential equations along a circle around the origin one finds:

$$\left\langle \Psi_1(e^{-2\pi i}z, e^{2\pi i}\,\overline{z})\; \Psi_2(0,\overline{0}) \right\rangle = e^{2\pi i\left(h_1-\bar{h}_1+h_2-\bar{h}_2\right)} \left\langle e^{2\pi i S_1}\, \Psi_1(z,\overline{z})\; e^{2\pi i S_2}\, \Psi_2(0,\overline{0}) \right\rangle \tag{40}$$

where Ψ_i abbreviates $\Psi_{(h_i,\bar{h}_i,\mathbf{k_i})}$ and S_i shortens $\delta_{h_i} - \delta_{\bar{h}_i}$. Here we introduced $\mathbf{k_i}$ anticipatorily as non-chiral Jordan level. As the notion Jordan level rests upon the Jordan block structure of the zero modes of the chiral symmetry algebra, we need a redefinition in terms of non-chiral representations, which will be given in section 4. Locality of the two-point function requires that

$$\left\langle \Psi_1(e^{-2\pi i}z, e^{2\pi i}\,\overline{z})\; \Psi_2(0,\overline{0}) \right\rangle = \left\langle \Psi_1(z,\overline{z})\; \Psi_2(0,\overline{0}) \right\rangle . \tag{41}$$

Comparing the right hand side of equation (40) with the right hand side of equation (41) reveals the following conditions for the two-point function to be local:

$$h_1 - \overline{h}_1 + h_2 - \overline{h}_2 \in \mathbb{Z}, \quad \left\langle S_1^n \Psi_1(z,\overline{z})\, S_2^m \Psi_2(0,\overline{0}) \right\rangle = 0 \quad \forall n,m \in \mathbb{Z}_{\geq 0} \quad n+m > 0. \tag{42}$$

Since this has to hold for arbitrary combinations of fields Ψ_1 and Ψ_2 the first condition must be devised even more strictly

$$h_i - \overline{h_i} \in \mathbb{Z}, \tag{43}$$

the second one can be shortened by

$$S_i \Psi_i = 0. \tag{44}$$

The scope of the conditions for a two-point function to be local can be enlarged to arbitrary n-point functions. By defining Ψ_2 as a suitable contour integral of the product of $n-1$ fields, every n-point function including Ψ_1 can be traced back to a two-point function $\langle \Psi_1\, \Psi_2 \rangle$.

4 Non-chiral local representations

Non-chiral irreducible representations can be obtained as diagonal tensor product of irreducible chiral representations [28]. This course of action fails in case of non-chiral indecomposable representations. A chiral indecomposable representation $\mathcal{R}_h$ of Jordan rank $r+1$ is generated by states $|h, k_i\rangle$, k_i running from r to null. The various tensor products included in $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ can be endowed with a gradation: Starting with the tensor product of those states with highest Jordan level, all other possible tensor products can be obtained by repeated application of δ_h and $\delta_{\bar{h}}$. The resulting structure is summarised in figure 1:

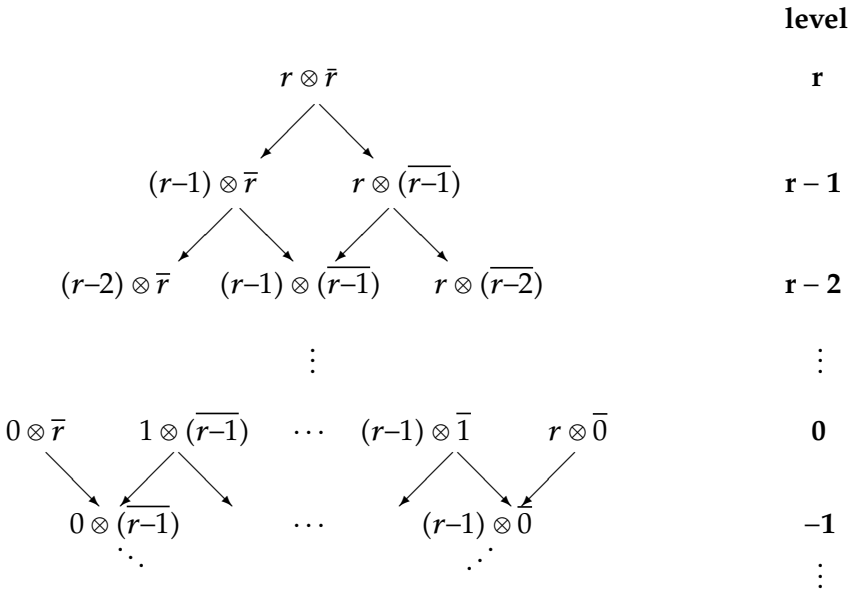


Figure 1: Gradation of $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$

Here the following abbreviations were introduced: An element of the product space $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ is uniquely denoted by its chiral and anti-chiral Jordan level

$$k \otimes \bar{k}' := |h, k\rangle \otimes |\bar{h}, \bar{k}'\rangle. \tag{45}$$

Let $\swarrow$ indicate that the state the arrow points at is an image of the state the arrow starts at under the action of δ_h. The symbol $\searrow$ does the same for the action of $\delta_{\bar{h}}$. Level $n = r - m$ is subsequently assigned to states that are obtained from $r \otimes \bar{r}$ by

m-fold descending, i.e. by m-fold application of delta operators. Starting with one state on level r, the number of states with level n increases gradually with decreasing $n \geq 0$. Level zero possesses the highest number of states, namely $r + 1$. For level $n \leq 0$ those states located on the fringes of the above graphic are deleted either by δ_h or by $\delta_{\bar{h}}$. Hence the number of states on level n decreases with decreasing $n \leq 0$:

$$\text{number of states in } \mathcal{R}_h \otimes \mathcal{R}_{\bar{h}} \text{ with level } n \;=\; r + 1 - |n|\,, \qquad |n| \leq r. \tag{46}$$

Due to the field-state isomorphism (22) equation (42) enforces that the action of the operator $S = \delta_h - \delta_{\bar{h}}$ has to vanish on all non-chiral states belonging to local theories. Figure 1 shows that generally the image of a tensor state under the action of δ_h differs from its image under $\delta_{\bar{h}}$. As a consequence states of the local non-chiral theory span only a subspace of the space generated by the diagonal tensor product of the chiral and anti-chiral Jordan cells [20]. This local subspace $\mathcal{L}_{h\bar{h}}$ can be constructed in two ways.

Firstly, we revisit the "quotient space construction" which Gaberdiel and Kausch introduced for the $c = -2$ local triplet theory [20] and enlarge the scope of their findings to theories whose chiral halves are of arbitrary rank. We would like to point out that a generalisation was achieved earlier but under a different viewpoint: Starting off at a single boundary condition Gaberdiel and Runkel constructed the local space of bulk states [24]. Their course of action holds for rational CFTs, logarithmic and non-logarithmic, and was shown to reproduce the known local bulk theory at $c = -2$ if applied to the $c_{1,2}$ triplet model. Our approach towards an enhanced quotient space construction differs from [24] inasmuch it does not use any information about boundaries.

Secondly, we introduce the "kernel construction" and compare the results of both methods.

4.1 Quotient space construction

Locality requires that the image of a local state under the action of δ_h has to equal the image of the same state under the action of $\delta_{\bar{h}}$ (42). The "quotient space construction" presented here approaches the problem by identifying both images modulo elements of a subspace $\mathcal{N}_{h\bar{h}}$:

$$\mathcal{L}_{h\bar{h}\,QSC} \equiv (\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}) \,/\, \mathcal{N}_{h\bar{h}}. \tag{47}$$

The chiral version of the $c = -2$ triplet theory possesses two indecomposable representations of rank two. For these cases, Gaberdiel and Kausch identified the

subrepresentation spanned by S acting on the tensor product of states with Jordan level one and the descendants of this state with respect to δ_h and $\delta_{\bar{h}}$ to be a minimal choice for $\mathcal{N}_{h\bar{h}}$ [20]:

$$\mathcal{N}_{h\bar{h}} = \left\{ (\delta_h)^p (\delta_{\bar{h}})^q S(1 \otimes \bar{1}) \right\}, \quad h = \bar{h} = 0, 1. \tag{48}$$

This result can be generalised to local theories whose chiral halves exhibit Jordan cells of arbitrary rank $r + 1$. For that purpose we repeat the procedure used to gain the gradation of $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ with a different starting point: S acting on the tensor product of those states with highest Jordan level. The structure of the resulting subrepresentation $S_{h\bar{h}}$ is depicted on the right hand side of the chart given below.

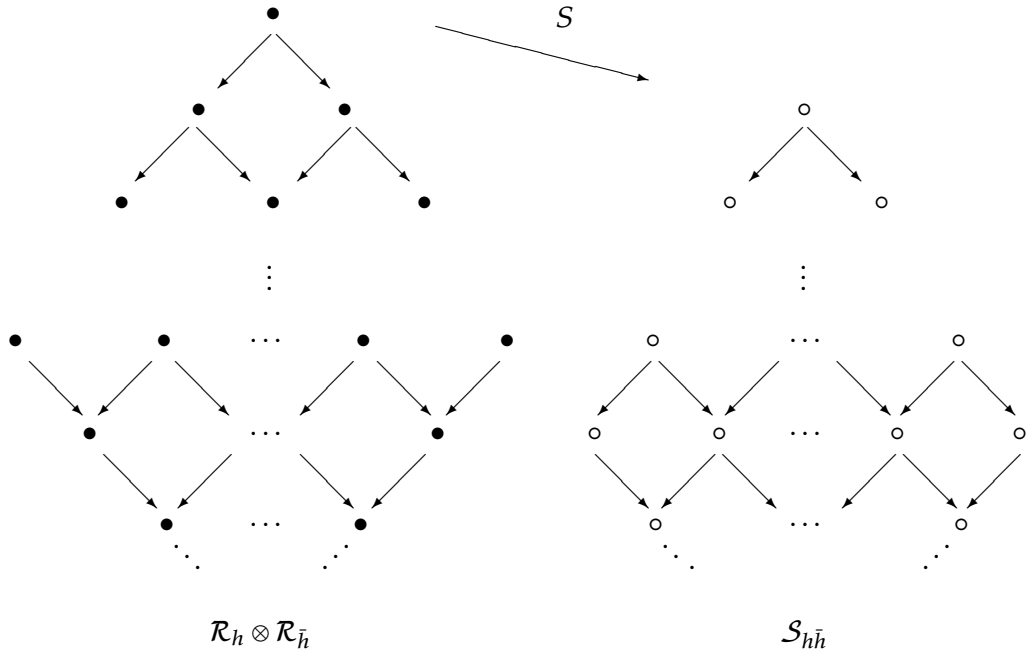


Figure 2: $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ versus $\mathcal{S}_{h\bar{h}}$

On the left hand side the gradation of $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ is recapitulated: Every • marks an element of the product space. As per construction the formation on the right hand side is generated by one state $S(r \otimes \bar{r}) = (r\text{–}1) \otimes \bar{r} - r \otimes (\overline{r\text{–}1})$ on level $r - 1$. This state and every state that emanates from it by application of delta operators are pictured as ∘. Counting the states $\in \mathcal{S}_{h\bar{h}}$ with level n yields

$$\text{number of states in } \mathcal{S}_{h\bar{h}} \text{ with level } n \;=\; r + 1 - |n + 1|, \qquad |n| \leq r - 1. \tag{49}$$

Level $n = -1$ accomodates the maximum number of states. Level $n \leq -1$ states that are located on the fringes vanish either under the action of δ_h or of $\delta_{\bar{h}}$.

Comparing the state content of both representations one finds: For $|n| \leq r - 1$, the difference of two adjacent level n states in $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ is element of $S_{h\bar{h}}$. Additionally, for level $n \leq -1$ the skirting states of both representations are pairwise identical:

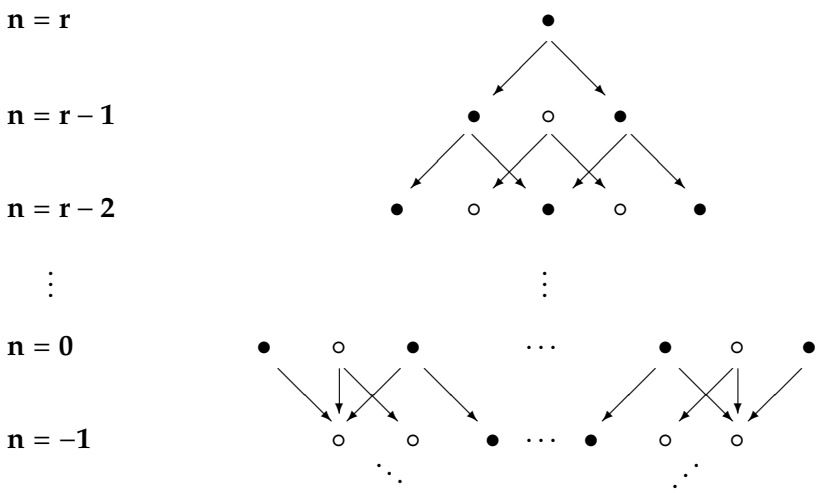


Figure 3: Comparison of $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ and $\mathcal{S}_{h\bar{h}}$

Hence dividing the subspace $\mathcal{S}_{h\bar{h}}$ out of $\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ accords with identification of all those states $\in \mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ that exhibit identical level:

$$(\delta_h)^{p_i}(\delta_{\bar{h}})^{q_i}\, r \otimes \bar{r} \sim (\delta_h)^{p_j}(\delta_{\bar{h}})^{q_j}\, r \otimes \bar{r} \qquad \forall\, i,j \,|\, p_i + q_i = p_j + q_j. \tag{50}$$

The action of δ_h on these equivalence classes equals that of $\delta_{\bar{h}}$. Thus indeed with $S_{h\bar{h}}$ we constructed a minimal choice for $\mathcal{N}_{h\bar{h}}$.

The equivalence classes in $(\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}})\,/\mathcal{S}_{h\bar{h}}$ are parameterised and arranged in order by their level. A non-chiral level n state shall be defined as a representative of the equivalence class with level n. As standard representative we choose the symmetric sum over all elements:

$$\big|h, \bar{h}, \mathbf{n}\big\rangle := \sum \text{level } n \text{ states}. \tag{51}$$

To avoid confusion with the Jordan level naming of a chiral state, we will use bold numbers to denote states of the non-chiral theory by their level. Since the equivalence classes with level $n \leq -1$ include representatives $\in \mathcal{S}_{h\bar{h}}$, they are entirely

removed from $(\mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}) / \mathcal{S}_{h\bar{h}}$, i. e. states $\in \mathcal{R}_{h\bar{h}}$ at level zero are annihilated by both delta operators.

4.2 Kernel construction

The "kernel construction" defines $\mathcal{L}_{h\bar{h}\,KC} \subset \mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ as the kernel of S. The key idea for determining the kernel of S is to use telescoping series. If we sum up all states $\in \mathcal{R}_h \otimes \mathcal{R}_{\bar{h}}$ with same level and act with S on it, every term except the first and the last cancels with either the preceding or suceeding term.

$$S\left(\sum \text{level } n \text{ states}\right) = \begin{cases} \delta_h\left(n \otimes \bar{r}\right) - \delta_{\bar{h}}\left(r \otimes \bar{n}\right) & \text{for } n \geq 1 \\ \delta_h\left(0 \otimes (r-n)\right) - \delta_{\bar{h}}\left((r-n) \otimes \bar{0}\right) & \text{for } n \leq 0. \end{cases} \tag{52}$$

As for level ≤ 0 the surviving terms vanish, we can conclude:

$$\mathcal{L}_{h\bar{h}\,KC} = \left\{ \sum \text{level } n \text{ states} \mid n \leq 0 \right\}. \tag{53}$$

4.3 Summary and discussion

In the preceding sections we presented two methods how to construct local indecomposable representations - the quotient space construction (QSC) and the kernel construction (KC). According to both methods $\mathcal{L}_{h\bar{h}}$ possesses the same rank as the chiral halves it is composed of:

$$\operatorname{rank}(\mathcal{L}_{h\bar{h}}) = \operatorname{rank}(\mathcal{R}_h) = \operatorname{rank}(\mathcal{R}_{\bar{h}}). \tag{54}$$

The state content of a local representation depends on the method used to construct it:

$$\begin{aligned} \mathcal{L}_{h\bar{h}\,QSC} &= \left\{ \sum \text{level } n \text{ states} \mid n \geq 0 \right\} \\ \mathcal{L}_{h\bar{h}\,KC} &= \left\{ \sum \text{level } n \text{ states} \mid n \leq 0 \right\} \end{aligned} \tag{55}$$

Above, specifications were made such that for QSC the term non-chiral Jordan level is in perfect accordance to the chiral Jordan level. For KC it is of avail to slightly adapt the definition of the non-chiral Jordan level: Redefining level n to be level $n + r$

$$n \mapsto n' = n + r$$

we achieve the familiar situation of non-negative integer values for the non-chiral Jordan level. Furthermore the redefinition guarantees that level zero states are annihilated by δ_h and $\delta_{\bar{h}}$:

$$\delta_h \left|h, \bar{h}, \mathbf{n}'\right\rangle_{KC} = \delta_{\bar{h}} \left|h, \bar{h}, \mathbf{n}'\right\rangle_{KC} = 0 \quad \text{for } \mathbf{n}' = \mathbf{0}. \tag{56}$$

Computable predictions in QSC may depend on the choice of the considered representative. The vanishing Shapovalov form of $|\mathbf{0}\rangle$ for example can be shown by means of the representatives $0 \otimes \bar{r}$ and $r \otimes \bar{0}$. However, rank $r+1$ theories with $r+1 > 1$ and r an even integer possess a level zero representative $\frac{r}{2} \otimes \frac{\bar{r}}{2}$. The corresponding Shapovalov form is

$$\left(\left\langle \tfrac{r}{2}\right| \otimes \left\langle \tfrac{\bar{r}}{2}\right|\right) \left(\left|\tfrac{r}{2}\right\rangle \otimes \left|\tfrac{\bar{r}}{2}\right\rangle\right) = \left\langle \tfrac{r}{2} \middle| \tfrac{r}{2} \right\rangle \left\langle \tfrac{\bar{r}}{2} \middle| \tfrac{\bar{r}}{2} \right\rangle \neq 0. \tag{57}$$

In a rank $r+1$ theory with $r+1 > 2$ and r an odd integer the same problem occurs but on level $\mathbf{n} = 1$: Such a theory possesses a level one representative $\frac{r+1}{2} \otimes \overline{\frac{r+1}{2}}$ that can easily be shown to contradict equation (38). The crucial point is that though representatives of QSC equivalence classes are as per construction equivalent with respect to the action of L_0 and $\bar{L}_0$ in other respects their equivalence is not guaranteed. Another example of this problematic QSC feature will be sketched in **Section 6**.

In every case is true: Representatives of QSC equivalence classes may be chosen such that observables in both formulations - QSC and KC - are identical. That is, even though $\mathcal{L}_{h\bar{h}\,QSC} \neq \mathcal{L}_{h\bar{h}\,KC}$ both representations are isomorphic and $\left|h, \bar{h}, \mathbf{n}\right\rangle_{QSC}$ is equivalent to $\left|h, \bar{h}, \mathbf{n}'\right\rangle_{KC}$. Hence, at this point it is both impossible and unnecessary to finally rule on the question wether one method has to be prefered.

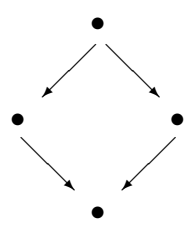
Anyhow, the known symplectic fermion realisation of the LCFT at $c = -2$ may be interpreted as a hint on the KC to be the more natural method. A symplectic fermion a la Zamolodchikov is a two-component fermionic field θ of spin zero [36]. The stress energy tensor of its free theory is given by $T(z) = \frac{1}{2}\epsilon^{\alpha\beta} : \partial\theta_\alpha \, \partial\theta_\beta : (z)$. The mode expansion of the component fields reads:

$$\theta_\alpha = \sum_{n \neq 0} \theta_{\alpha,n} z^{-n} + \theta_{\alpha,0} \log(z) + \xi_\alpha. \tag{58}$$

The ξ's are Grassmann numbers and act as creation operators for the chiral logarithmic partner of the identity:

$$|h = 0, k = 1\rangle = -\frac{1}{2}\epsilon^{\alpha\beta}\, \xi_\alpha \, \xi_\beta \, |h = 0, k = 0\rangle =: \xi^\alpha \, \xi_\beta \, |0\rangle. \tag{59}$$

Of course an analogue identity holds for the anti-chiral half. Therewith we can give explicit expressions for the sum of all states $\in \mathcal{R}_0 \otimes \mathcal{R}_{\bar{0}}$ with equal level n:



$$\xi^\alpha \xi_\beta |0\rangle \otimes \bar{\xi}^{\bar{\alpha}} \bar{\xi}_{\bar{\beta}} |\bar{0}\rangle = \xi^\alpha \xi_\beta \bar{\xi}^{\bar{\alpha}} \bar{\xi}_{\bar{\beta}} |0\rangle \otimes \bar{0}\rangle$$

$$\xi^\alpha \xi_\beta |0\rangle \otimes |\bar{0}\rangle + |0\rangle \otimes \bar{\xi}^{\bar{\alpha}} \bar{\xi}_{\bar{\beta}} |\bar{0}\rangle = \left(\xi^\alpha \xi_\beta + \bar{\xi}^{\bar{\alpha}} \bar{\xi}_{\bar{\beta}}\right) |0\rangle \otimes |\bar{0}\rangle$$

$$|0\rangle \otimes |\bar{0}\rangle$$

It is possible to choose a basis such that $\xi^\alpha = \bar{\xi}^{\bar{\alpha}}$, i. e. the state $|\mathbf{n} = \mathbf{1}\rangle$ vanishes due to the nilpotency of ξ. This coincides with the prediction of the kernel construction.

5 Correlation functions

The prominent role that correlation functions play in CFTs results from two facts: On the one hand, they are related to observables and therefore represent a connection between theory and accessible experimental data. On the other hand, they are considered fundamental from a pure theoretical point of view: As shown in [37], a CFT is completely constituted if all correlation functions are known. Given the two- and three-point functions of the fundamental fields, all other amplitudes can actually be derived from these. Furthermore, the consistency conditions of all amplitudes can be traced back to those obeyed by the four-point functions.

The calculation of correlation functions in LCFTs holds two major difficulties that do not arise in case of non-logarithmic CFTs. Both have their origin in the non-diagonal action of the generators of the chiral symmetry algebra. The off-diagonal contribution enters the global conformal Ward identities in the shape of an inhomogeneity and results in the aforementioned challenges:

Firstly the identification of the generic structure of chiral correlation functions compatible to global conformal invariance is remarkably hindered. A hierarchical solution scheme for the inhomogeneous Ward identities allows to explore the texture of the subset of correlators which contain chiral fields residing in indecomposable representations whose irreducible subrepresentation corresponds to a proper primary field. For these cases, it is possible to fix the generic structure of n-point functions up to structure functions of $n-3$ $SL(2,\mathbb{C})$ invariant crossratios, however only within sets of other correlators. Possible extensions of the hierarchical solution scheme to pre-logarithmic fields and non-quasi-primaries are discussed in [4] and [10].

Secondly, correlation functions of an LCFT do not generally factorise into chiral and anti-chiral parts. This is also an immediate consequence of the inhomogeneous Ward identities which is mirrored by the fact that only those correlators are factorisable that solve Ward identities with vanishing inhomogeneity. Gurarie pointed out that even amplitudes not explicitly involving logarithmic fields do not necessesarily fall in this category [2]. Two attemps have been made to adapt the knowledge about chiral correlation functions for non-chiral ones. Flohr provided a rule of thumb, how to generalise known chiral sets of correlation functions to local sets by replacing all emerging variables $z_i - z_j =: z_{ij}$ by $|z_{ij}|^2$ [35]. This approach preserves the full generality of the chiral sets but obscures the interrelationship between chiral, anti-chiral and non-chiral amplitudes. Its validity is proven in Appendix A. Gaberdiel and Kausch suceeded in constructing a consistent set of amplitudes for the local theory at $c = -2$ [20]. As their course of action rests crucially upon

model specific information like the operator product expansion of the fundamental fields, it cannot be transfered to the generic case.

Our proceeding will be as follows: We first recapitulate the nessessary assumptions under which generic chiral n-point functions can be calculated and briefly describe the hierachical solution scheme for these cases. Subsequently, we summarise the generic structure of the n-point functions found this way. In section 5.2 we will give a short proof for the statement that a non-chiral correlation function factorises if and only if the inhomogeneity of the Ward Identities vanishes for the chiral correlators it is composed of. Finally we present an ansatz built solely out of quantities enclosed in the chiral and anti-chiral sets of n-point correlators that allows the construction of local n-point amplitudes.

Correlators of fields residing in the chiral (anti-chiral) half of an LCFT will be named chiral, anti-chiral respectively. Even though "chiral" intrinsically describes propagation properties of fields, we prefer this term to the adjunct holomorphic which is often chosen to indicate that a function only depends on the formal variable z but not on $\bar{z}$.

5.1 Chiral correlation functions

Within this thesis, we will consider correlation functions of fields $\Psi_{(h_i,k_i)}(z_i)$ from Jordan cells whith $\Psi_{(h_i,0)}(z_i)$ a proper primary. The logarithmic partner fields are assumed to be quasi-primary, i. e. $L_n\Psi_{(h_i,k_i)}(0)\,|0\rangle = 0 \quad \forall\, n \geq 1$.

LCFT correlation functions are invariant under the global conformal group. This stipulates the generic texture of n-point correlators up to structure functions of $n-3$ $SL(2,\mathbb{C})$ invariant crossratios that are a priori undetermined. Evaluating the identity (34) for $q = -1, 0, 1$ exhibits the inhomogeneous global conformal Ward identities:

$$O_q \langle \ldots \rangle = -\sum_{i=1}^{n} z_i^q (q+1)\delta_{h_i} \langle \ldots \rangle \,, \qquad \text{for } q \in \{-1,0,1\}\,. \tag{60}$$

From equation (60) follows immediately that correlation functions containing logarithmic fields cannot be determined independently: Due to the action of δ_{h_i}, the generic structure of an n-point function including fields $\Psi_{(h_i,k_i)}(z_i)$ can only be specified within a framework of other n-point functions containing fields $\Psi_{(h_i,k_i-1)}(z_i)$. Given a set of n conformal weights h_i with $r(h_i) = r$ there exists a hierarchy of r^n n-point functions with $s = 0, \ldots, n$ logarithmic fields displaying varying Jordan levels $k_i = 1, \ldots, r-1$. The number of different correlation functions of a set is actually

reduced: As discussed in **Section 2.2.3** a correlator is non-zero only, if the sum over the Jordan levels of the logarithmic fields it contains equals at minimum $r-1$.

The identity (38) serves as starting point for a recursive construction of solutions of the GCWIs (60). For total Jordan level $K = r-1$, the GCWIs are homogeneous and can be solved as known from ordinary CFT. Successive increase of the total Jordan level yields differential equations for $\langle k_1 \dots k_n \rangle$ with the inhomogeneity $\sum_i \delta_{h_i} \langle k_1 \dots k_n \rangle$ determined in foregoing steps of the recursion. To improve lucidity we wil subsequently shorten the naming of the correlators:

$$\langle k_1 \dots k_n \rangle := \left\langle \Psi_{(h_1,k_1)}(z_1) \dots \Psi_{(h_n,k_n)}(z_n) \right\rangle . \tag{61}$$

One finds that n-point correlators which contain fields of rank r Jordan cells posses the generic form:

$$\langle k_1 \dots k_n \rangle = \prod_{i<j} (z_i - z_j)^{\mu_{ij}} \sum_{G=0}^{l^{max}} F^G_{\{q\}}(x_a) \cdot P_G(l_{mn}), \qquad l_{mn} := \ln(z_{mn}) \tag{62}$$

where P_G denotes a sum over monomials of degree G:

$$P_G = \sum_{\alpha | g(\alpha)=G} c_\alpha \prod_{i=1}^{j} (l_{m_i n_i})^{g_{\alpha_i}} =: \sum_\alpha c_\alpha p_\alpha, \qquad \sum_{i=1}^{j} g_{\alpha_i} = g(\alpha). \tag{63}$$

The constraint of global conformal invariance (60) connects a coefficient c_α multiplying a monomial p_α in a correlator A to the coefficient c_β which multiplies a monomial p_β in a correlator $\sum_i \delta_{h_i} A$, with p_β being the image of p_α under the action of O_0. Four-point or higher correlation functions that exhibit logarithmic fields at every vertex may feature polynomials $K_G \subset P_G$, whose multiplicities are special in the following sense: They are not cross-linked to any coefficients in other n-point functions of the set. Linkage to correlators with lower total Jordan level is canceled if K_G resides in the kernel of the operator $O := (O_0, O_1)$ [10]. No linkage to higher correlators of the set has to be required seperatly. As a consequence kernel terms K_G may only arise in the highest correlator of a set, i. e. in the correlator where all inserted fields are of maximum Jordan level $k_n = r-1$.

The occurring monomials p_α are subject to selection rules [35]: Logarithms in the correlators stem from contractions of logarithmic fields. Hence, only such logarithms may arise whose indices refer to positions of fields with Jordan level $k \geq 1$ within a correlator. To see this and beyond it how the possible powers with which l_{mn} may occur are determined it is instructive to use a graphical representation first described by Flohr (2003):

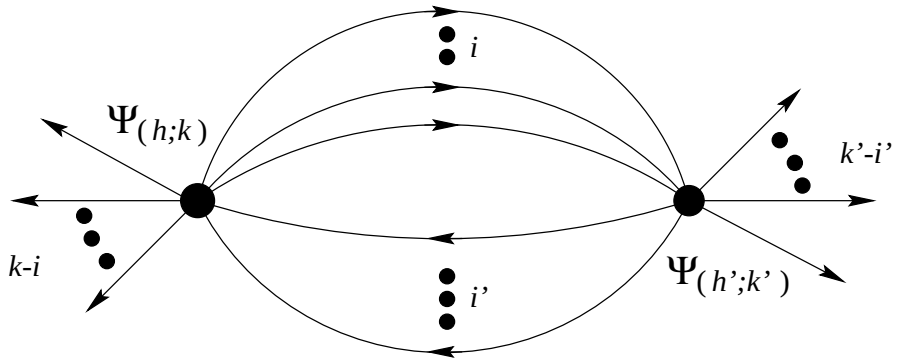


Figure 4: Graphical representation of contractions of logarithmic fields

Each field $\Psi_{(h,k)}$ is depicted by a vertex with k outgoing lines. These legs are to be connected with other vertices, whereas three rules have to be observed:

1. A vertex $\Psi_{(h,k)}$ may receive at most $r-1$ incoming lines, where r is the rank of the LCFT.
2. The source and the destination vertex have to be different.
3. Connections to primary fields are not possible.

Every connection between two vertices m and n represents a logarithm l_{mn}. As shown above a correlation function of a rank r LCFT vanishes if its total Jordan level $K = \sum_i k_i$ is lower than $r-1$ (c. f. equation (38)). In terms of the proposed graphical representation this means that leastwise $r-1$ legs have to remain uncontracted. If the number of legs is greater or equal $r-1$ the correlation function will essentially be a sum over all possible graphs where at most $\sum_i k_i - (r-1)$ legs are linked to arbitrary vertices.

The outlined restrictions ruling the logarithmic terms can be summed up in the following selection rules:

[S1] Only such logarithms may arise whose indices refer to positions of fields with Jordan level $k \geq 1$.

[S2] The total logarithmic degree G in equation (62) is bounded above as follows:

$$G \leq K - r + 1 =: l^{max}.$$

[S3] Each index m_i may arise at most $r-1$ times within one monomial.

Let $F^G_{\{q\}}(x_a)$ denominate a family of functions, which solely depend on $n-3$ anharmonic ratios x_a. The subscript $\{q\}$ denotes a set of n indices q_i which take integer

values between zero and k_i :

$$F^G_{\{q\}} := F^G_{q_1 q_2 \ldots q_n}\,, \qquad q_i \in \{0, 1 \ldots k_i\}\,. \tag{64}$$

For fixed superscript index G, only those combinations $\{q\}$ emerge that fulfil

$$\sum_i q_i + G = K. \tag{65}$$

Coefficient functions satisfying $\sum q_i = r - 1$ have to be identified due to the cluster decomposition property [10]. They will be refered to as $F^{l^{max}}$. For $n \leq 3$ $F^G_{\{q\}}$ does not depend on the values q_i but only on $\sum_i q_i$ [4].

The structure functions $F_{\{q\}}(x)$ may be decomposed in conformal blocks $\mathcal{F}^i_{\{q\}}(x)$ which represent the internal propagators:

$$F_{\{q\}}(x) = \sum_i \mathcal{F}^i_{\{q\}}(x).$$

During the main part of this paper this decomposition will not play any role for the presented argumentation. To keep things simple we will abstain from making it explicitly where it is not necessary.

The exponents μ_{ij} in equation (62) solve

$$\sum_{i \neq j} \mu_{ij} = -2h_j. \tag{66}$$

So far we have stated properties of chiral correlation functions only. It is clear that analogues propositions hold for anti-chiral correlators.

5.2 Assembling local amplitudes from the chiral sets

We can now bring the original query into sharper focus. Non-chiral amplitudes shall be obtained by multiplying suitable chiral and anti-chiral amplitudes. Therefore we have to revisit the constraints of locality (42). As in the frame of this thesis only the case $h_i = \bar{h}_i$ is considered, the first condition does not cause concern. Using equation (60) the second condition can be restated as follows:

$$-\sum_{i=1}^{n} S_i \langle \Psi_1 \ldots \Psi_n \rangle = \left(O_0 - \bar{O}_0\right) \langle \Psi_1 \ldots \Psi_n \rangle = 0. \tag{67}$$

In non-logarithmic conformal field theories, the non-chiral amplitudes $\mathbf{A}$ can be achieved by multiplying the chiral and anti-chiral amplitudes A and $\bar{A}$:

$$\mathbf{A} = A\bar{A}. \tag{68}$$

In LCFTs, factorisation is contradictory to the locality constraint except for correlators satisfying homogeneous Ward identities:

$$\begin{aligned}(O_0 - \overline{O}_0)\mathbf{A} &= (O_0 A)\bar{A} - A(\bar{O}_0\bar{A}) \\ &= -\left(\sum_i \delta_{h_i} A\right)\bar{A} + A\left(\sum_i \delta_{\bar{h}_i}\bar{A}\right) = 0 \;\Leftrightarrow\; \sum_i \delta_{h_i} A = \sum_i \delta_{\bar{h}_i}\bar{A} \equiv 0\end{aligned} \tag{69}$$

where the first identity follows from the fact, that the operators O_0 and $\bar{O}_0$ act as derivatives with respect to z ($\bar{z}$ respectively) on the function space, i. e. $\bar{O}_0$ acting on the chiral amplitude does not yield a contribution and vice versa. The second identity arises out of the chiral amplitudes satisfying the GCWIs. The last step is based on the fact, that the maximum logarithmic degree of $\delta_{h_i} A$ is reduced by one compared to the maximum logarithmic degree of A.

The given argumentation is not affected if the conformal block decomposition of the amplitudes is taken into account, i. e. if equation (68) is substituted by

$$\mathbf{A} = \sum_{qp} \mathcal{K}_{qp} A_q \bar{A}_p,$$

where A_q denotes the contribution of a conformal block q to A. As the conformal blocks do only depend on the crossratios, adjustment of their linear combination can not cancel the mismatch of logarithmic powers in equation (69).

Subsequently, we present an ansatz that admits the construction of generic local n-point functions out of the known chiral correlators:

$$\left\langle \mathbf{k_1} \dots \mathbf{k_n} \right\rangle = \left\langle k_1 \dots k_n \right\rangle \left\langle \overline{k_1 \dots k_n} \right\rangle |_{\text{selection rules}} + GOL, \tag{70}$$

where GOL stands for guarantor of locality and lives up to its name by providing the desired behavior of $\langle \mathbf{k_1} \dots \mathbf{k_n} \rangle$ under the action of $O_0 - \overline{O}_0$. This fixes $(O_0 - \overline{O}_0)\, GOL$ as follows:

$$(O_0 - \overline{O}_0)GOL = \sum_{i=1}^{n} \left[\left(\delta_{h_i} \left\langle k_1 \dots k_n \right\rangle \right) \left\langle \overline{k_1 \dots k_n} \right\rangle - \left\langle k_1 \dots k_n \right\rangle \left(\delta_{\bar{h}_i} \left\langle \overline{k_1 \dots k_n} \right\rangle \right) \right]. \tag{71}$$

The contribution of

$$\left\langle k_1 \dots k_n \right\rangle \left\langle \overline{k_1 \dots k_n} \right\rangle = \prod_{i<j} |z_{ij}|^{2\mu_{ij}} \left[\sum_{g=0}^{l^{max}} F^{G}_{\{q\}}(x) \cdot P_G \left(l_{m_s n_s} \right) \right] \left[\sum_{g=0}^{l^{max}} \bar{F}^{\bar{G}}_{\{\bar{q}\}}(\bar{x}) \cdot \bar{P}_{\bar{G}} \left(\overline{l_{m_r n_r}} \right) \right]$$

in equation (70) is constricted to terms satifying selection rules. The selection rules for the arising logarithmic terms in chiral correlators have been resumed in **Section 5.1**. Generalising them to the non-chiral case is straightforward: [S1] is trivially fulfilled. The highest logarithmic degree to appear in a correlator was shown to depend on its total Jordan level and the rank of the theory, [S2]. As demonstrated in section 4 the rank of the non-chiral theory equals the rank of the chiral halves it is composed of. Furthermore, as per construction the total Jordan level of the left hand side of equation (70) equals the total Jordan level of the chiral correlators $\left\langle k_1 \dots k_n \right\rangle$ and $\left\langle \overline{k_1 \dots k_n} \right\rangle$, i. e.

$$g + \bar{g} \leq l^{max}, \qquad l^{max}_{non-chiral} = l^{max}_{chiral} = l^{max}_{anti-chiral}. \tag{72}$$

The highest multiplicity for one index to appear within a monomial was stated to solely depend on the rank of the theory, [S3]. Thus, according to the aforementioned reasoning, it can be adopted from the chiral case as its stands.

Therewith the left hand side of the ansatz (70) can in principle be calculated. For that purpose, we have to expand the product of the chiral correlators, implement [S2] and [S3] and add an expansion of *GOL*. The latter is given as follows:

$$GOL = \prod_{i<j} |z_{ij}|^{2\mu_{ij}} \sum_{G=0}^{l^{max}} \sum_{\alpha,\beta} c_{\alpha\beta} \prod_{s=1}^{t} \prod_{r=1}^{u} \left(l_{m_s n_s} \right)^{g_{\alpha_s}} \left(\overline{l_{m_r n_r}} \right)^{\bar{g}_{\beta_r}}, \ \sum_{s=1}^{t} \sum_{r=1}^{u} (g_{\alpha_s} + \bar{g}_{\beta_r}) = G. \tag{73}$$

It is clear that coefficients $c_{\alpha\beta}$ depend on the chiral and anti-chiral structure functions, i. e.

$$c_{\alpha\beta} = c_{\alpha\beta} \left(F_{\{q\}}, \bar{F}_{\{\bar{q}\}} \right).$$

For further convenience, we will subsequently use the naming convention introduced in equation (63):

$$p_\alpha := \prod_{s=1}^{t} (l_{m_s n_s})^{g_{\alpha_s}}, \qquad \sum_{s=1}^{t} g_{\alpha_s} = g(\alpha) =: g_\alpha. \tag{74}$$

Herewith the expansion of *GOL* (73) reduces to the form:

$$GOL = \prod_{i<j} |z_{ij}|^{2\mu_{ij}} \sum_{G=0}^{l^{max}} \sum_{\alpha,\beta} c_{\alpha\beta}\, p_\alpha \bar{p}_\beta. \tag{75}$$

Of course, the logarithmic terms of *GOL* are subordinated to [S1] - [S3], too. Further restrictions on the structure of *GOL* arise from its claimed behavior under the action of $(O_0 - \overline{O_0})$: Equation (71) changes sign under complex conjugation. It follows that

$$GOL = \overline{GOL} \quad \Rightarrow \quad c_{\alpha\beta} = \overline{c_{\beta\alpha}}, \text{ in particular } c_{\alpha\alpha} \in \mathbb{R}. \tag{76}$$

In addition the coefficients $c_{\alpha\beta}$ are coupled to linear combinations of the structure functions $F^G_{\{q\}}(x)$ and $\bar{F}^{\bar{G}}_{\{\bar{q}\}}(\bar{x})$ by two sets of constraints. The first set arises out of the condition (71). The second emanates from the logarithmic identities governing the assembly of local monomials: Monodromy invariance of equation (70) enforces arguments of emerging logarithms to be real, i. e. logarithmic terms have to be of the shape

$$\begin{aligned}
\mathbf{p}_\alpha(|l_{m_s n_s}|^2) :&= \prod_{s=1}^{t} (|l_{m_s n_s}|^2)^{g_s} \\
&= \prod_{s=1}^{t} \left(l_{m_s n_s} + \overline{l_{m_s n_s}} \right)^{g_s} \\
&= \prod_{s=1}^{t} \left(\sum_{i=0}^{g_s} \binom{g_s}{i} \left(l_{m_s n_s} \right)^i \left(\overline{l_{m_s n_s}} \right)^{g_s - i} \right).
\end{aligned} \tag{77}$$

Hence, coefficients of terms $\prod_{s=1}^{t} \left(l_{m_s n_s} \right)^i \left(\overline{l_{m_s n_s}} \right)^{g_s - i}$ with $0 \leq i \leq g_s$ are fixed up to an overall factor. These coefficients are proportional to $F^G_{\{q\}}(x) \bar{F}^{\bar{G}'}_{\{\bar{q}'\}}(\bar{x})$ if a monomial stems from $\left\langle k_1 \dots k_n \right\rangle \left\langle \overline{k_1 \dots k_n} \right\rangle$ or else given by $c_{\alpha\beta}$, which establishes the aforementioned coupling. Recalculating one finds, that every solution of the resulting set of constraints solves equation (71).

Without loss of generality the free choice of an overall factor multiplying equation (77) can be absorbed in the factors of contributions $\prod_{s=1}^{t} \left(l_{m_s n_s} \right)^{g_s} \left(\overline{l_{m_s n_s}} \right)^0$. This lightens our ansatz (70) to

$$\langle \mathbf{k_1} \dots \mathbf{k_n} \rangle = \prod_{i<j} \left| z_{ij} \right|^{2\mu_{ij}} \sum_{G=0}^{l^{max}} \sum_{\alpha} \underbrace{\left(\bar{F}^{\bar{0}}_{\{\bar{q}'\}} F^G_{\{q\}} c_\alpha + c_{\alpha 0} \right)}_{=: \mathbf{C}_\alpha} \mathbf{p}_\alpha. \tag{78}$$

Within this generic approach the monodromy properties of the coefficients $\mathbf{C}_\alpha$ cannot be completely explored. The structure functions $\bar{F}_{\{\bar{q}'\}}(\bar{x})$, $F_{\{q\}}(x)$ in equation

(78) are linear combinations of conformal blocks:

$$\bar{F}_{\{\bar{q}'\}}(\bar{x}) = \sum_i \bar{\mathcal{F}}^i_{\{\bar{q}'\}}(\bar{x}), \quad F_{\{q\}}(x) = \sum_j \mathcal{F}^j_{\{q\}}(x), \quad \bar{F}_{\{\bar{q}'\}} F_{\{q\}} = \sum_{ij} \mathcal{K}_{ij} \bar{\mathcal{F}}^i_{\{\bar{q}'\}} \mathcal{F}^j_{\{q\}}. \tag{79}$$

Enforcing monodromy invariance of the amplitude (78) determines the coefficients $\mathcal{K}_{ij}$, a task that cannot be performed within the generality achieved here.

It is worth pointing out that the claims asserted so far suffice to guarantee all coefficients $\mathbf{C}_\alpha$ being real: The constraint (77) in particular demands that the $i = 0$ term and the $i = g_s$ term arise with the same multiplicity. The coefficient of the former can easily be shown to equal $\bar{F}^{\bar{G}}_{\{\bar{q}\}} F^0_{\{q'\}} c_\alpha + c_{0\alpha}$. Using equation (76) we can conclude:

$$\bar{F}^{\bar{G}}_{\{\bar{q}\}} F^0_{\{q'\}} c_\alpha + c_{0\alpha} = \overline{\mathbf{C}_\alpha} = \mathbf{C}_\alpha \quad \Rightarrow \mathbf{C}_\alpha \in \mathbb{R}. \tag{80}$$

Reimplementing the GCWIs establishes dependencies between coefficients $\mathbf{C}_\alpha$ of different local correlators of a set. The occurring cross-linkage of coefficients is in perfect analogy to the chiral case: $\mathbf{C}_\alpha$ emerging in a correlator $\mathbf{A}$ is connected to $\mathbf{C}_\beta$ in a correlator $\sum_i \delta_{h_i} \mathbf{A} = \sum_i \delta_{\bar{h}_i} \mathbf{A}$ if $\mathbf{p}_\beta$ is the image of $\mathbf{p}_\alpha$ under the action of O_0 as well as under the action of $\bar{O}_0$. It follows that the obtained solution for a generic chiral set of correlation functions, obeying locality and global conformal invariance is not unique. The number of degrees of freedom a local correlator possess equals the number of those the corresponding chiral correlator shows. This matches with the predictions of Flohr's substitution method [35]. Anyhow it is astonishing as one could have expected that implementing the condition of locality would confine the number of free parameters. Investigating duality [28, 38] of the obtained generic correlation functions yields that even this constraint does not reduce the number of free parameters any further.

5.3 Summary and discussion

Non-uniqueness of the solution

Generic local sets of correlators cannot uniquely be determined by virtue of locality, duality and global conformal invariance. The most general form of *GOL* preserves the number of free parameters a set of correlation functions exhibits. The minimal choice for *GOL* fulfilling the constraints (71) and (77) is build up of mixed terms only, i. e. it does not exhibit any contributions $p_\alpha \bar{p}_\beta$ with $g_\alpha = 0$ or $\bar{g}_\beta = 0$. As a

consequence, $c_{\alpha 0}$ in equation (78) vanishes identically for all α. In this case, reimplementing the GCWIs fixes all but one conformal block of a set. We are confronted with a situation very similar to non-logarithmic CFT: Although in the LCFT case a whole hierarchy of n-point functions emanates from a set of n conformal weights $h_i = \bar{h}_i$ the structure of each correlator is fixed up to one shared structure function F which solely depends on $n-3$ crossratios and the conformal weights h_i but not on $\mathbf{k_i}$.

Consistency check

Our method allows us to connect the results of [10] to the results of [20]. According to [10] the rank two chiral set of four-point functions with $h_i = 0 \,\forall\, i$ reads

$$\langle 1000 \rangle = F_{1000}\,, \tag{81a}$$

$$\langle 1100 \rangle = \mathcal{P}_{S_2}\left\{ \tfrac{1}{2} F_{1100} - l_{12} F_{1000} \right\}, \tag{81b}$$

$$\langle 1110 \rangle = \mathcal{P}_{S_3}\left\{ \tfrac{1}{6} F_{1110} + (\tfrac{1}{2} l_{23} - l_{12}) F_{0110} + \left[l_{12} l_{23} - \tfrac{1}{2} l_{12}^2 \right] F_{1000} \right\}, \tag{81c}$$

$$\begin{aligned} \langle 1111 \rangle = \mathcal{P}_{S_4}\Big\{ & \tfrac{1}{24} F_{1111} + (\tfrac{1}{6} l_{23} - \tfrac{1}{3} l_{12}) F_{0111} + \\ & \left[\tfrac{1}{2}(l_{14} l_{23} - l_{12} l_{34}) + (l_{12} l_{23} - \tfrac{1}{2} l_{24} l_{23}) - \tfrac{1}{4} l_{12}^2 \right] F_{0011} + \\ & \left[\tfrac{1}{2} l_{12}^2 l_{34} + \tfrac{1}{3} l_{12} l_{23} l_{13} - l_{12} l_{23} l_{34} \right] F_{1000} \Big\}\,. \end{aligned} \tag{81d}$$

Missing correlators of the set can be obtained by permutation of the inserted fields. $\mathcal{P}_{S_x}$ denotes the sum over all permutations of indices generated by the group S_x. As all four fields posses the same conformal weight $h_i = h_j$ it follows from the associativity of the fusionalgebra that

$$F_{q_1 q_2 q_3 q_4}(x) = F_{q_{\sigma(1)} q_{\sigma(2)} q_{\sigma(3)} q_{\sigma(4)}}(x) \qquad \forall \sigma \in S_4, \tag{82}$$

i. e. $\mathcal{P}_{S_x}$ does only affect the indices of logarithmic terms l_{ij}. The symmetry (82) admits to identify all $F_{\mathbf{q}}$ with equal $\sum q_i = p$ and to abbreviate them by the p-th capital letter of the latin alphabet. As $\Psi_{h=0,k=0}$ denotes the identity field the amplitudes (81a-c) are identical to the one-, two- and three-point functions of the field $\Psi_{h=0,\,k=1}$. Consequentially, in the following expressions, A, B and C are constant, $D = D(x)$.

We can now arrange the equations (81) more clearly:

$$\langle 1000\rangle = A\,, \tag{83a}$$

$$\langle 1100\rangle = B - 2A\,(l_{12})\,, \tag{83b}$$

$$\langle 1110\rangle = C - B\,(l_{12}+l_{13}+l_{23}) + A\big[2\,(l_{12}l_{23}+l_{13}l_{23}+l_{12}l_{13}) - \left(l_{12}^2+l_{13}^2+l_{23}^2\right)\big]\,, \tag{83c}$$

$$\langle 1111\rangle = D - \frac{2}{3}C\,{*}{-}{*} + 2B\big[(2\kappa+1)\,{*}{-}{*}\,{*}{-}{*} - \kappa\,{*}{-}{*}{-}{*} + \kappa{*}{=}{*}\big] + 2A\big[{*}{=}{*}\,{*}{-}{*} + \underbrace{{*}{-}{*}{-}{*}} - {*}{-}{*}{-}{*}{-}{*}\big]\,. \tag{83d}$$

The correlation function (83d) exhibits a kernel term K_G whose multiplicity κ is not fixed by global conformal invariance. Moreover in support of clarity we chose a graphical notation to depict the logarithmic terms in this correlator. Reading the diagrams is straightforward: Each position denotes a variable indexvalue. The symbol filling a position designates the variable which carries the index. We will use $*$ for z and $\triangle$ for $|z|^2$. A line between two symbols of the same kind indicates the logarithm of the symbollically represented variable. Each graph stands for the sum over all those identifications of variable indexvalues with numerical indexvalues which provide expressions that are not equivalent, e. g.

$$\triangle{-}\triangle\triangle{-}\triangle = \mathbf{l}_{12}\mathbf{l}_{34} + \mathbf{l}_{13}\mathbf{l}_{24} + \mathbf{l}_{14}\mathbf{l}_{23}, \qquad \mathbf{l}_{ij} := \ln(\left|z_{ij}\right|^2).$$

Combining local amplitudes according to the method described above yields:

$$\langle \mathbf{1000}\rangle = \mathbf{C}_{AA}\,, \tag{84a}$$

$$\langle \mathbf{1100}\rangle = \mathbf{C}_{BB} - 2\mathbf{C}_{BA}\,(\mathbf{l}_{12})\,, \tag{84b}$$

$$\langle \mathbf{1110}\rangle = \mathbf{C}_{CC} - \mathbf{C}_{CB}\,(\mathbf{l}_{12}+\mathbf{l}_{13}+\mathbf{l}_{23}) + \mathbf{C}_{CA}\big[2\,(\mathbf{l}_{12}\mathbf{l}_{23}+\mathbf{l}_{13}\mathbf{l}_{23}+\mathbf{l}_{12}\mathbf{l}_{13}) - \left(\mathbf{l}_{12}^2+\mathbf{l}_{13}^2+\mathbf{l}_{23}^2\right)\big]\,, \tag{84c}$$

$$\langle \mathbf{1111}\rangle = \mathbf{C}_{DD} - \frac{2}{3}\mathbf{C}_{DC}\triangle{-}\triangle + 2\mathbf{C}_{DB}\big[(2\lambda+1)\triangle{-}\triangle\triangle{-}\triangle - \lambda\triangle{-}\triangle{-}\triangle + \lambda\triangle{=}\triangle\big] + 2\mathbf{C}_{DA}\big[\triangle{=}\triangle\triangle{-}\triangle + \underbrace{\triangle{-}\triangle{-}\triangle} - \triangle{-}\triangle{-}\triangle{-}\triangle\big]\,, \tag{84d}$$

where $\mathbf{C}_{XY} = \bar{X}Y + c_{xy}$ with c_{xy} being a complex number corresponding to $c_{\alpha 0}$ in formula (78). The reimplementation of the GCWIs enforces the identification all coefficients $\mathbf{C}_{XY}$ with same second index:

$$\mathbf{C}_{XY} \equiv \mathbf{C}_Y \;\forall\; X. \tag{85}$$

It is worth noticing that in the course of assembling local amplitudes the arbitrary multiplicities κ and $\bar{\kappa}$ of the chiral kernel term are set to zero. The kernel term

$$\lambda \mathbf{K}_G = \lambda \, (2 \, \triangle\!-\!\triangle\triangle\!-\!\triangle \; - \; \triangle\!-\!\triangle\!-\!\triangle \; + \; \triangle\!=\!\triangle)$$

in the correlator (84d) is exclusively composed of contributions of *GOL*.

We can now resume: The equations (84) constitute a generic set of correlators containing fields from a rank two reducible representation with $h = \bar{h} = 0$. One explicitly known representative of this case is given through the set of n-point functions $\left\langle \prod_n \Psi_{h_n=0,\bar{h}_n=0,\mathbf{k_n}=\mathbf{1}} \right\rangle$, $n = 1, \ldots 4$ of the local LCFT at $c = -2$ [20]. Except for an overall sign of the correlator (84d) both sets - the generic and the concrete - are consistent: For the local triplet theory, the undetermined coefficients of the generic set take the values

$$\begin{aligned} \mathbf{C}_A &= \mathcal{C}_0 = 1, & \mathbf{C}_B &= -8 \ln(2), \\ \mathbf{C}_C &= 48 \ln^2(2), & \mathbf{C}_D &= -256 \ln^3(2). \end{aligned}$$

Furthermore the kernel multiplicity λ is fixed to $-\frac{1}{2}$ in virtue of the operator product expansion.

6 Conclusions

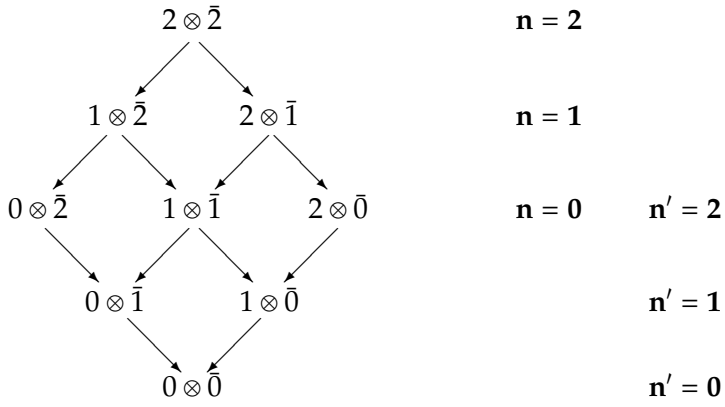
Exploiting the conformal symmetry allows first steps towards the construction of generic local LCFTs. The generality of our approach is confined by the following assumptions: Indecomposable representations are considered only with respect to the Virasoro zero mode. Irreducible subrepresentations are assumed to correspond to proper primaries, logarithmic partners to quasi-primaries. Furthermore only the diagonal case $h_i = \bar{h}_i$ is regarded.

For future work, generalising our findings by releasing these assumptions would be an interesting task: As mentioned before the scope of section 5 can possibly be extended to pre-logarithmic fields and non-quasi-primaries [4, 10]. Remarkably, it seems that even abdicating the $h_i = \bar{h}_i$ condition does not seriously damage the proposed method for the construction of generic local correlation functions out of generic chiral ones.

A matter of particular interest could be to investigate the local space of states for a theory whose chiral halves violate the assumption of quasi-primarity of the logarithmic partner fields. In case of chiral Jordan cells which contain at least one logarithmic field $\Psi_{h,k}$ ($\bar{\Psi}_{\bar{h},\bar{k}}$ respectively) with the property

$$\exists\, n > 0 : \lim_{z\to 0} L_n \, \Psi_{h,k}(z)\,|0\rangle \neq 0$$

the equivalence of QSC and KC is broken. This shall be illustrated by means of a rank three non-chiral representation where the chiral Jordan level one fields are not quasi-primary.



According to both methods non-quasi-primarity of the chiral level k states induces non-quasi-primarity of the non-chiral level $\mathbf{k}$ state (here $\mathbf{n} = \mathbf{1}$, $\mathbf{n}' = \mathbf{1}$ respectively). Pursuant to QSC additionally states with lower level are affected (here the level $\mathbf{n} = \mathbf{0}$ representative $1 \otimes \bar{1}$). By contrast after KC the non-quasi-primarity is passed on states with higher level (here the $\mathbf{n}' = \mathbf{2}$ state $0 \otimes \bar{2}+1 \otimes \bar{1}+2 \otimes \bar{0}$). Two cases of the sketched scenario can be distiguinshed:

1. Rank $r \geq 3$: In KC non-quasi-primarity (NQP) of a chiral state with Jordan level k ($\bar{k}$ respectively) encroaches upon all local states with $\mathbf{n}' \leq k$. In QSC the spread of NQP can be supressed by suitable choice of the considered representative unless the non-quasi primary logarithmic partner is the field with highest Jordan rank.
2. Rank $r = 2$: In KC the only possible non-quasi-primary is identical to the field with highest Jordan level, i. e. the NQP does not spread. In QSC every level zero representatives is affected by NQP. We deal with a situation that could be called converse quasi-primarity as the behaviour under L_0 and $\bar{L}_0$ equals the behaviour of a proper primary but for every representative

   $$|\mathbf{n} = \mathbf{0}\rangle_\alpha = (1 \otimes \bar{0} + 0 \otimes \bar{1}) + \alpha(1 \otimes \bar{1} - 0 \otimes \bar{1}),\ \alpha \in \mathbb{C}$$

   exists a positiv Virasoro mode such that

   $$\left(\bar{L}_n\right)^i (L_n)^j |\mathbf{n} = 0\rangle_\alpha \neq 0, \quad i, j \in \{0, 1\}, \quad i + j \neq 0.$$

Exploring the first case might bring light to a problem we have already sketched in section 4.3. It seems desirable to understand how the uncertainty can be mastered which enters computable predications in the QSC formulation through the arbitrary but non-equivalent choice of the regarded representative. Actually, a promising possibility to cope with the spreading of NQP in the framework of QSC might be to extend $\mathcal{N}_{h\bar{h}}$. For an example see Appendix C.

Somehow or other, the broken QSC–KC equivalence in case of NQP provides the opportunity to treat them as competing models. On the basis of an explicitly known realisation exhibiting the claimed properties it could become possible to determine wether the predicted spread of NQP is reasonable. And finally the comparison of the explicit realisation with the predications of QSC and KC could resolve the question if one of the proposed methods can be adapted to the NQP case.

A Proving the rule of thumb

In [35] a rule of thumb was proposed, how to obtain a non-chiral version out of known chiral correlators by replacing

1. every $z_i - z_j =: z_{ij}$ by $|z_{ij}|^2$ and
2. every structure function $F_{\{q\}}(x)$ by $\tilde{F}_{\{q\}}(|x|^2)$.

To give a proof that this method yields local, GCWI consistent sets of correlation functions one has to show that if $p_\beta(\ln(z_{mn}))$ is the image of $p_\alpha(\ln(z_{mn}))$ under the action of O_0, $\mathbf{p}_\beta\left(\ln|z_{mn}|^2\right)$ is the image of $\mathbf{p}_\alpha\left(\ln|z_{mn}|^2\right)$ under the action of O_0 as well as under the action of $\overline{O_0}$:

$$O_0\, p_\alpha(l_{mn}) \;=\; p_\beta(l_{mn}) \quad \Leftrightarrow \quad O_0\, \mathbf{p}_\alpha(\mathbf{l}_{mn}) \;=\; \mathbf{p}_\beta(\mathbf{l}_{mn}) = \overline{O_0}\, \mathbf{p}_\alpha(\mathbf{l}_{mn})\,.$$

It is convenient to consider at first monomials p_α of the shape $(l_{mn})^{g_\alpha}$. In this case O_0 acting on p_α yields

$$O_0\ (l_{mn})^{g_\alpha} = g_\alpha\ (l_{mn})^{\,g_\alpha - 1} =: p_\beta\,(l_{mn})\,.$$

O_0 acting on $\mathbf{p}_\alpha(\mathbf{l}_{mn}) = (\mathbf{l}_{mn})^{g_\alpha}$ provides

$$\begin{aligned}
O_0\ \left(\ln|z_{mn}|^2\right)^{g_\alpha} &= O_0\ \left(l_{mn} + \overline{l_{mn}}\right)^{g_\alpha} \\
&= O_0 \sum_{k=0}^{g_\alpha} \binom{g_\alpha}{k} \left(l_{mn}\right)^k\ \left(\overline{l_{mn}}\right)^{g_\alpha - k} \\
&= \sum_{k=0}^{g_\alpha - 1} \frac{g_\alpha!(g_\alpha - 1)}{(g_\alpha - k)!\ k!}\ \left(l_{mn}\right)^k\ \left(\overline{l_{mn}}\right)^{g_\alpha - 1 - k} \qquad (86) \\
&= \sum_{k=0}^{g_\beta = g_\alpha - 1} g_\alpha \binom{g_\beta}{k} \left(l_{mn}\right)^k\ \left(\overline{l_{mn}}\right)^{g_\beta - k} \\
&= \mathbf{p}_\beta\,(\mathbf{l}_{mn})\,. \qquad q.e.d.
\end{aligned}$$

Line three to five of equation (86) are invariant under complex conjugation and so is $\mathbf{p}_\alpha(\ln|z_{mn}|^2)$. It follows that the action of $\overline{O_0}$ equals the action of O_0 when applied on $\mathbf{p}_\alpha(\ln|z_{mn}|^2)$.

As any monomial can be decoded as product of monomials of the above shape, it is straightforward to upgrade the given proof to arbitrary p_α.

$$p_\alpha(l_{m_i n_i}) = \prod_{i=1}^{j} (l_{m_i n_i})^{g_{\alpha_i}} =: \prod_{i=1}^{j} p_{\alpha_i}(l_{m_i n_i}), \qquad \sum_{i=1}^{j} g_{\alpha_i} = g_\alpha$$

According to the product rule O_0 acting on $p_\alpha(l_{m_i n_i})$ yields a sum over j summands, each of them containing $j-1$ unchanged factors $p_{\alpha_i}(l_{m_i n_i})$ and one factor $p_{\beta_i}(l_{m_i n_i})$. Hence the generic case is put down to the special case treated above.

B Assembling local correlation functions: An example

In this section the proceeding proposed to construct local correlators from the chiral sets shall be illustrated. We choose the easiest non-trivial example: The assembly of a non-chiral correlator of four fields residing in a $r = 2$ Jordan cell all but one possesing level one. Furthermore we will consider the special case, where an additional symmetry improves lucidity: If all four fields possess the same conformal weight $h_i = h_j$, all structure functions $F_{q_1 q_2 q_3 q_4}(x)$ are invariant under permutation of the indices q_i (cf. equation (82)). In this case the generic chiral four-point functions calculated in [10] can be formulated by means of a graphical notation which was fragmentarily introduced in section 5.3 and will subsequently be explored in some more detail.

B.1 Graphical notation proper

To depict the logarithmic terms in $h_i = h_j$-type n-point functions it is convenient to use a graphical notation which improves clarity of the expressions and facilitates dealing with them. Reading the diagrams is straightforward: Each position denotes a variable indexvalue. The symbol filling a position designates the variable which carries the index. We will use $*$ for z, $\circ$ for $\overline{z}$ and $\triangle$ for $|z|^2$. A line between two symbols of the same kind indicates the logarithm of the difference of the symbollically represented variables.

$$*{=}\circledast{-}\circledast{-}* := \sum_{i,j,k,m} l_{ij}^2 \cdot \bar{l}_{jk} \cdot l_{km} \ , \quad \bar{l}_{jk} := \ln(\bar{z}_j - \bar{z}_k) \tag{87}$$

Each graph stands for the sum over all identifications of variable indexvalues with numerical indexvalues providing inequivalent expressions. Of course only those

numerical values have to be taken into account that are conformable with [S1]. What a graph denotes is therefore in multiple respects depending on the context it is used in:

The number of monomials in the l_{ij} a graph represents is depending on the number of fields with Jordan level $k \geq 1$.

> For example the graph $*\text{–}*$ in the chiral four-point function $\langle 1110 \rangle$ represents the sum over the three monomials l_{12}, l_{13} and l_{23}. The same graph in the chiral four-point function $\langle 1111 \rangle$ denotes the sum over six monomials, namely $l_{12}, l_{13}, l_{14}, l_{23}, l_{24}$ and l_{34}.

Mostly the context makes clear, which expression is symbolised by a graph. Where it is not we will attach an index that designates the numer of incorporated monomials.

Which concrete numerical values the indices can take depends on the positions the fields with Jordan level $k \geq 1$ occupy within a correlator.

> For example the graph $*\text{–}*$ in the chiral four-point function $\langle 1110 \rangle$ represents the sum over the three monomials l_{12}, l_{13} and l_{23}. The same graph in the chiral four-point function $\langle 1101 \rangle$ denotes the sum over the monomials l_{12}, l_{14} and l_{24}.

One advantage of the graphical representation suggested is that the calculation of sums over correlators as well as of products of correlation functions reduces to concise combinatorial considerations. In the following calculations we will benefit from this properties.

Using the specified notation the subset of $r = 2$ chiral four-point functions with $h_i = h_j$ reads

$$\prod_{i<j} z_{ij}^{-\mu_{ij}} \langle 1000 \rangle = A \tag{88a}$$

$$\prod_{i<j} z_{ij}^{-\mu_{ij}} \langle 1100 \rangle = B - 2A \; *\text{–}*_1 \tag{88b}$$

$$\prod_{i<j} z_{ij}^{-\mu_{ij}} \langle 1110 \rangle = C - B \; *\text{–}*_3 + A(2 \; *\text{–}*\text{–}*_3 \; - \; *=*_3) \tag{88c}$$

$$\begin{aligned} \prod_{i<j} z_{ij}^{-\mu_{ij}} \langle 1111 \rangle = D - \frac{2}{3} C \; *\text{–}*_6 + 2B((2\alpha + 1) \; *\text{–}* \; *\text{–}*_3 - \alpha \; *\text{–}*\text{–}*_{12} \\ + \alpha *=*_6) + 2A(*=* \; *\text{–}*_6 + \underbrace{*\text{–}*\text{–}*}_{4} - *\text{–}*\text{–}*\text{–}*_{12}) \end{aligned} \tag{88d}$$

Missing correlators can be obtained by permutation. This changes only the meaning of the graphs. The coefficient functions which are invariant under permutation are

subject to the naming convention chosen in section 5.3: All coefficients $F_{\mathbf{q}}$ with equal $\sum q_i = p$ are identified and abbreviated them by the p-th capital letter of the latin alphabet.

B.2 Explicit calculation

In the following the non-chiral local correlator $\langle \mathbf{1110} \rangle$ is explicitly calculated. Pursuant to the ansatz (70) the local correlator adds up to:

$$\langle \mathbf{1110} \rangle = \left\langle 1110 \right\rangle \left\langle \overline{1110} \right\rangle + GOL. \tag{89}$$

Using equation (88c) the contribution of the product of chiral correlators can easily be calculated:

$$\begin{aligned}\prod_{i<j} |z_{ij}|^{-2\mu_{ij}} \langle 1110 \rangle \left\langle \overline{1110} \right\rangle = \bar{C}C \; - \; \bar{C}B \, *\!\!-\!\!*_3 \; - C\bar{B} \, \circ\!\!-\!\!\circ_3 \; + \bar{C}A \, (2 \, *\!\!-\!\!*\!\!-\!\!*_3 \; - *\!=\!*_3) \; + \\ \bar{A}C \, (2 \, \circ\!\!-\!\!\circ\!\!-\!\!\circ_3 \; - \circ\!=\!\circ_3) \; + \; \bar{B}B(*\!\!-\!\!*_3 \cdot \circ\!\!-\!\!\circ_3) \; + \; \mathcal{O}(G{=}3).\end{aligned} \tag{90}$$

Here, logarithmic terms of higher than quadratic order are omitted to make allowance for selection rule [S2]. Computing the product $*\!\!-\!\!*_3 \cdot \circ\!\!-\!\!\circ_3$ is a matter of simple combinatorics:

$$*\!\!-\!\!*_3 \cdot \circ\!\!-\!\!\circ_3 = \circ\!\!-\!\!\circledast\!\!-\!\!*_6 \; + \; \circledast\!=\!\circledast_3. \tag{91}$$

In the next step we have to analyze GOL which can be expanded as follows:

$$\begin{aligned}\prod_{i<j} |z_{ij}|^{-2\mu_{ij}} \, GOL = c_{00} \; + \; c_{01} \, *\!\!-\!\!*_3 \; + c_{10} \, \circ\!\!-\!\!\circ_3 \; + c_{11} \, \circ\!\!-\!\!\circledast\!\!-\!\!*_6 \; + c'_{11} \circledast\!=\!\circledast_3 \\ c_{02} \, *\!\!-\!\!*\!\!-\!\!*_3 \; + c_{03} *\!=\!*_3 \; + \; c_{20} \, \circ\!\!-\!\!\circ\!\!-\!\!\circ_3 \; + c_{30} \circ\!=\!\circ_3.\end{aligned} \tag{92}$$

Using the structure of GOL (76) we can lighten the above expression by the following redefinitions:

$$\begin{aligned} c_{00} &=: c_1 \in \mathbb{R} & c_{01} = \overline{c_{10}} &=: c_2 \\ c_{11} &=: c_3 \in \mathbb{R} & c_{02} = \overline{c_{20}} &=: c_4 \\ c'_{11} &=: c_5 \in \mathbb{R} & c_{03} = \overline{c_{30}} &=: c_6 \end{aligned}$$

Furthermore $\left(O_0 - \overline{O_0}\right) GOL$ is determined by equation (71). In order to calculate the left hand side of this relation it is of avail to clarify the action of O_0 and $\overline{O_0}$ on

the space of polynomials in the logarithms $S_{log} := \mathbb{C}\left[l_{ij}, \bar{l}_{ij}\right]$. The simple behaviour can be summarized as follows [10]:

$$O_0 : S_{log} \mapsto S_{log} \quad O_0\left(l_{i_1 j_1} \dots l_{i_n j_n}\right) = \sum_{k=1}^{n} l_{i_1 j_1} \dots l_{i_{k-1} j_{k-1}} l_{i_{k+1} j_{k+1}} \dots l_{i_n j_n}. \tag{93}$$

In terms of a graphically denoted polynomial $\mathcal{G} \in S_{log}$ that means: $O_0\,\mathcal{G}$ equals the sum over all possibilities to remove one line between two symbols $*$, $\overline{O_0}\,\mathcal{G}$ acts analogous on lines between symbols $\circ$. Therewith the left hand side of equation (71) is given through

$$\left(O_0 - \overline{O_0}\right) GOL = 3c_2 - 3\bar{c_2} + (-2c_3 - c_5 + 2c_4 + 2c_6)\ *\!\!-\!\!*_3 - (-2c_3 - c_5 + 2\bar{c}_4 + 2\bar{c}_6)\ \circ\!\!-\!\!\circ_3 \tag{94}$$

The right hand side reads:

$$\begin{aligned}
&\sum_{i=1}^{3}\left[\left(\delta_{h_i}\left\langle 1110\right\rangle\right)\left\langle\overline{1110}\right\rangle - \left\langle 1110\right\rangle\left(\delta_{\bar{h}_i}\left\langle\overline{1110}\right\rangle\right)\right] = \\
&\quad = \left(3B - 2A\ *\!\!-\!\!*_3\right)\left(C - B\ \circ\!\!-\!\!\circ_3 + A(2\ \circ\!\!-\!\!\circ\!\!-\!\!\circ_3\ -\ \circ\!\!=\!\!\circ_3)\right) - \text{c. c.} \\
&\quad = 3C\bar{B} - 3\bar{C}B + \left(-2\bar{C}A + 3B\bar{B}\right)\ *\!\!-\!\!*_3 - \left(-2C\bar{A} + 3B\bar{B}\right)\ \circ\!\!-\!\!\circ_3 + \mathcal{O}(G = 2).
\end{aligned} \tag{95}$$

Via comparison of coefficients we can conclude that identity (71) imposes the following conditions on the coefficients:

$$3c_2 - 3\bar{c_2} = 3C\bar{B} - 3\bar{C}B\,, \tag{96}$$

$$-2c_3 - c_5 + 2c_4 + 2c_6 = -2\bar{C}A + 3B\bar{B}\,. \tag{97}$$

Inserting the expressions (90) and (92) in the ansatz (89) provides:

$$\begin{aligned}
\prod_{i<j}\left|z_{ij}\right|^{-2\mu_{ij}}\left\langle\mathbf{1110}\right\rangle = (\bar{C}C + c_1) + \\
&(-\bar{C}B + c_2)\ *\!\!-\!\!*_3 + \left(-C\bar{B} + \bar{c_2}\right)\ \circ\!\!-\!\!\circ_3 + \\
&\left(2\bar{C}A + c_4\right)\ *\!\!-\!\!*\!\!-\!\!*_3 + \left(2C\bar{A} + \bar{c_4}\right)\ \circ\!\!-\!\!\circ\!\!-\!\!\circ_3 + \left(B\bar{B} + c_3\right)\ (\circ\!\!-\!\!\circledast\!\!-\!\!*_6) + \\
&\left(-\bar{C}A + c_6\right)\ *\!\!=\!\!*_3 + \left(-C\bar{A} + \bar{c_6}\right)\ \circ\!\!=\!\!\circ_3 + \left(B\bar{B} + c_5\right)\ \circledast\!\!=\!\!\circledast_3
\end{aligned} \tag{98}$$

The arrangement of equation (98) is not deliberately chosen. Its logarithmic terms have to be combined to logarithms of real arguments. The adherent condition (77) possesses an elegant formulation in terms of graphs $\mathcal{G}$:

$$\mathcal{G}(\triangle) = \mathcal{G}(*) + \mathcal{G}(\circ) + \text{all pairwise distinct possibilities to compose } \mathcal{G} \text{ from } *\!\!-\!\!* \text{ and } \circ\!\!-\!\!\circ. \tag{99}$$

The example given below shall demonstrate this rule:

$$\triangle\!-\!\triangle\!-\!\triangle_3 = *\!-\!*\!-\!*_3 \;+\; \circ\!-\!\circ\!-\!\circ_3 \;+\; *\!-\!\circledast\!-\!\circ_3 \;+\; \circ\!-\!\circledast\!-\!*_3.$$

In case of graphs that contain more than one line between two positions a additional rule has to be observed: Mixed multilines have to rise with multipicities, e. g.

$$\triangle\!=\!\triangle\triangle\!-\!\triangle \;=\; *\!=\!* *\!-\!* \;+\; \circ\!=\!\circ\;\circ\!-\!\circ \;+\; *\!=\!*\;\circ\!-\!\circ \;+\; \circ\!=\!\circ\;*\!-\!* \;+\; 2\circledast\!=\!\circledast\;*\!-\!* \;+\; 2\circledast\!=\!\circledast\;\circ\!-\!\circ\,.$$

Therewith the following sets of constraints can line-by-line be excerpted from equation (98):

$$-\bar{C}B + c_2 = -C\bar{B} + \bar{c}_2 \tag{100}$$

$$2\bar{C}A + c_4 = 2C\bar{A} + \bar{c}_4 = B\bar{B} + c_3 \tag{101}$$

$$-\bar{C}A + c_6 = -C\bar{A} + \bar{c}_6 = \tfrac{1}{2}\left(B\bar{B} + c_5\right). \tag{102}$$

One can see immediately: The conditions (96) and (100) coincide. Moreover constraint (97) is automatically satisfied if the equations (101) and (102) are fulfilled. Introducing the abbreviations

$$\mathbf{C}_{CC} := \bar{C}C + c_1\,, \quad -\mathbf{C}_{CB} := -\bar{C}B + c_2\,, \quad \mathbf{C}_{CA} := \bar{C}A + \tfrac{1}{2}c_4\,, \tag{103}$$

we can finally state the local correlator to be

$$\prod_{i<j} |z_{ij}|^{-\mu_{ij}} \langle \mathbf{1110} \rangle = \mathbf{C}_{CC} - B\triangle\!-\!\triangle_3 + A(2\triangle\!-\!\triangle\!-\!\triangle_3 \;-\; \triangle\!=\!\triangle_3). \tag{104}$$

For $h_i = 0,\ \forall\ i = 1,\ldots 4$ it is consistent with equation (84c).

C Cut off the spreading of NQP in QSC

The spreading of NQP in the framework of QSC might possibly be prevented by extending $\mathcal{N}_{h\bar{h}}$. As an example serves the $c = -2$ non-chiral representation with conformal weight $h = \bar{h} = 1$. On the chiral level of consideration we find the following structures:

$$L_0\ \phi = \phi + \psi\,, \qquad L_0\ \psi = \psi\,,$$
$$L_1\ \phi = -\xi\,, \qquad L_{-1}\ \xi = \psi\,,$$

i. e. ϕ and ψ span a rank two Jordan cell with respect to L_0. The logarithimc partner field ϕ is not a quasi-primary. It can be generated from the state ξ at level zero through the action of L_{-1}. The QSC formalism determines the non-chiral equivalent of the logarithmic partner to be

$$\mathbf{\Phi} = \phi \otimes \bar{\phi}. \tag{105}$$

The level zero equivalence class $\mathbf{\Psi}$ is represented by

$$\psi \otimes \bar{\phi} \sim \phi \otimes \bar{\psi}. \tag{106}$$

The encroaching of NQP upon the level $\mathbf{n} = 0$ state $\mathbf{\Psi}$ becomes apparent in the non-trivial action of L_1 on its representatives:

$$L_1 \ (\phi \otimes \bar{\psi}) = \overline{L_1} \ (\psi \otimes \bar{\phi}) \neq 0. \tag{107}$$

As far as locality is concerned there is no reason not to define the non-chiral irreducible representation $\mathbf{\Xi}$ as direct tensor product

$$\mathbf{\Xi} = \xi \otimes \bar{\xi}. \tag{108}$$

But if we choose the alternative definition

$$\mathbf{\Xi} = L_1\mathbf{\Phi} \ \wedge \ \mathbf{\Xi} = \overline{L_1}\mathbf{\Phi}, \tag{109}$$

we are forced to identify $\xi \otimes \bar{\phi} \sim \phi \otimes \bar{\xi}$. As a consequence $\mathcal{N}_{1\bar{1}}$ obtains an additional element: $\xi \otimes \bar{\Phi} - \Phi \otimes \bar{\xi}$. The non-vanishing images (107) of $\mathbf{\Psi}$ under the action of L_1 and $\overline{L_1}$ are therewith automatically $\in \mathcal{N}_{1\bar{1}}$ and the property of $\mathbf{\Psi}$ to be a proper-primary is restated.

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Acknowledgements

I wish to thank my supervisor PD Michael Flohr for his guidance throughout the last year. Initiating this work and accompanying all of its stages with helpful ideas and suggestions he always let me benefit from his great experience on the topic.

I'm also grateful to Prof. Frahm for kindly accepting to referee this thesis.

Many thanks to the cordial members of the ITP. I would especially like to thank my office mates Kirsten Vogeler and Hendrik Adorf. They always lend a ready ear, an incisive mind and patient empathy to support and challenge me and my project.

Thanks to Anne for being outspoken, Guillaume for ensuring my personal safety and to Holger for his support at the initial moments of this thesis. I'm grateful to Johannes, Andre, Achim, Kalle and Micha for great coffe breaks, their sense of humor and encouragement. A special gratitude is due to Martin Paech whose editing suggestions contributed to the final copy.